CARLEMAN ESTIMATES AND LOCAL UNIQUE CONTINUATION

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1. Introduction

These notes originated from reading seminars held in autumn 2019—with Vaibhav Jena, Alex McGill, and Dongbing Zha—on the classical Carleman estimates from the microlocal viewpoint. Its contents are primarily based on parts of the textbook *Carleman Inequalities: An Introduction and More* [3], written by Nicolas Lerner. The form of the Carleman estimates (and the associated local unique continuation result) presented here is mainly attributed to Hörmander [2, Chapter 28], and applies to a general class of differential operators.

Though we intend for the notes to be self-contained, we also aim to organize the material such that it minimizes the required background in microlocal analysis and pseudodifferential operators. While some engagement with microlocal analysis is unavoidable, here we restrict our coverage to only the parts that are necessary for our Carleman estimates—most notably the sharp Gårding inequality. In order to better streamline discussions, the proofs of essential results from microlocal analysis are given later in the Appendix.

Technical note: The Carleman estimates in these notes apply to operators with real principal symbols, and the proofs make use of the sharp Gårding inequality. Another variant of these estimates uses the Fefferman-Phong inequality instead and is also applicable to operators with complex principal symbols, under an additional assumption of principal normality. As a result, the choice of material presented here involves some tradeoffs; while the present version is extendedible to coefficients with less regularity and to systems of PDEs, the alternative version applies to a wider class of (only scalar-valued) operators.

Version note: The latest version of these notes is now designed to apply the sharp Gårding (rather than Fefferman-Phong) inequality, as this path involves less technical machinery and covers more situations of practical interest. Furthermore, corrections were made to the statement of the sharp Gårding (formerly Fefferman-Phong) inequality, and an appendix was added to address its proof. This required some non-trivial reorganization of the material—in particular, the main discussions now encompass general λ-parametrized symbols and pseudodifferential operators, as opposed merely to polynomials and differential operators.

2. Unique Continuation and Carleman Estimates

In this section, we discuss the local unique continuation problem for linear partial differential equations. We also formulate (local) Carleman estimates in an abstract manner, and we discuss their role as a key tool for solving unique continuation problems.

2.1. **The Main Setting.**
The initial task is to describe the setting within which we will discuss the unique continuation problem. This is expressed via the following definitions and assumptions, which we will adopt throughout these notes.

Definition 2.1. *In accordance with standard conventions:*

- *We use ∇ to denote the usual (Euclidean) gradient on $\mathbb{R}^n$.*
- *We also define the operator $D := -i\nabla$.*

In addition, for any multi-index $I = (I_1 \dots I_N)$ and $\xi \in \mathbb{R}^n$, we set

$$\nabla_I := \partial_{I_1} \dots \partial_{I_N}, \qquad D_I := (-i)^N \partial_{I_1} \dots \partial_{I_N}, \qquad \xi_I := \xi_{I_1} \cdots \xi_{I_N}. \tag{2.1}$$

Assumption 2.2. *Let Ω be an open subset of $\mathbb{R}^n$, and fix a hypersurface $\Sigma \subseteq \Omega$ given as*

$$\Sigma := \{x \in \Omega \mid \rho(x) = 0\}, \tag{2.2}$$

for some $\rho \in C^\infty(\Omega; \mathbb{R})$ such that

$$\nabla\rho(x) \neq 0, \qquad x \in \Sigma. \tag{2.3}$$

In addition, we fix a point $x_0 \in \Sigma$.

Remark 2.3. *In particular, all of our developments will be purely local in nature, thus our requirement in* (2.2) *that Σ is a level set does not result in any loss of generality.*

Remark 2.4. *Notice that ρ implictly defines an orientation of Σ, since $\nabla\rho$ is normal to Σ. In particular, we can think of ρ as selecting the side of Σ corresponding to $\rho > 0$.*

Assumption 2.5. *Let $\mathcal{P}$ be a linear partial differential operator of order m on Ω,*

$$\mathcal{P}\phi := \sum_{|I| \leq m} p^I D_I, \tag{2.4}$$

with $p^I \in C^\infty(\Omega; \mathbb{C})$ for each multi-index I. Moreover, let $\mathcal{P}_0$ denote the principal part of $\mathcal{P}$:

$$\mathcal{P}_0 := \sum_{|I| = m} p^I D_I. \tag{2.5}$$

Remark 2.6. *One could weaken Assumption 2.5 so that the coefficients p^I have less regularity. However, we avoid doing this here in order to simplify the exposition.*

We now recall the principal symbol of $\mathcal{P}$, which carries most of the essential information about $\mathcal{P}$ that we will require in the upcoming analysis:

Assumption 2.7. *The* principal symbol *of $\mathcal{P}$ is the function*

$$p \in C^\infty(\Omega \times \mathbb{R}^n; \mathbb{C}), \qquad p(x, \xi) := \sum_{|I| = m} p^I(x)\, \xi_I. \tag{2.6}$$

Remark 2.8. *By default, we will treat p as a homogeneous polynomial of n real variables ξ. However, in many situations, we will find it useful to implicitly extend p as a polynomial of complex variables, that is, as a function $p : \Omega \times \mathbb{C}^n \to \mathbb{C}$.*

We conclude by recalling some special types of differential operators:

Definition 2.9. *$\mathcal{P}$ is* elliptic *at x_0 iff $p(x_0, \xi) \neq 0$ for any $\xi \in \mathbb{R}^n \setminus \{0\}$.*

Definition 2.10. *Σ is* non-characteristic *at x_0 with respect to $\mathcal{P}$ iff*

$$p(x_0, d\rho(x_0)) \neq 0. \tag{2.7}$$

Definition 2.11. *Σ is* strictly hyperbolic *at x_0 with respect to $\mathcal{P}$ iff:*

- *Σ is non-characteristic at x_0 with respect to $\mathcal{P}$.*
- *If $\xi \in \mathbb{R}^n$ and $\xi \wedge d\rho(x_0) \neq 0$, then the polynomial*

$$f_{q,x_0,\xi} : \mathbb{C} \to \mathbb{C}, \qquad f_{q,x_0,\xi}(\sigma) := p(x_0, \xi + \sigma\, d\rho(x_0)) \tag{2.8}$$

has only simple (i.e., distinct) and real roots.

2.2. **Carleman Estimates.** The main tool, and the main topic of these notes, is a class of weighted integral inequalities known as Carleman estimates, which we describe below.

First, we formulate the class of weight functions that can be used for our estimates:

Definition 2.12. *$\phi \in C^\infty(\Omega; \mathbb{R})$ is a* Carleman weight *for ρ at x_0 iff for any neighborhood $U \subset\subset \Omega$ of x_0, there exist a neighborhood $V \subseteq U$ of x_0 and $\delta > 0$ such that:*

- *$\phi(x) > \phi(x_0)$ on the region $\{x \in V \mid \rho(x) > 0\}$.*
- *The following property holds:*

$$\{x \in \bar{V} \mid \rho(x) \geq 0,\ \phi(x) - \phi(x_0) \leq \delta\} \subseteq V. \tag{2.9}$$

Remark 2.13. *Roughly, Definition 2.12 states that the level sets $\phi - \phi(x_0) = \epsilon$, for small $\epsilon > 0$, must both extend into $\rho > 0$ near x_0 and exit $\rho > 0$ through Σ near x_0.*

Remark 2.14. *Note that by replacing ϕ with $\phi - \phi(x_0)$ in Definition 2.12, we can additionally assuming, without any loss of generality, that $\phi(x_0) = 0$.*

While Definition 2.12 is all that is required for our Carleman estimates, we will work with a specific family of Carleman weights for our upcoming main result:

Proposition 2.15. *For any $\mu > 0$, the function $\phi_\mu \in C^\infty(\Omega; \mathbb{R})$, given by*

$$\begin{aligned} \phi_\mu(x) := {} & \nabla\rho(x_0) \cdot (x - x_0) + \tfrac{1}{2}(x - x_0) \cdot \nabla^2\rho(x_0) \cdot (x - x_0) \\ & - \mu[\nabla\rho(x_0) \cdot (x - x_0)]^2 + \tfrac{1}{2\mu}|x - x_0|^2, \end{aligned} \tag{2.10}$$

is a Carleman weight for ρ at x_0.

Proof. First, note that (2.10) immediately yields $\phi_\mu(x_0) = 0$. Given U as in Definition 2.12, we let $V := B(x_0, \varepsilon)$ be an open ball about x_0, with radius $\varepsilon > 0$ small enough so that

$$V \subseteq U, \qquad \bar{V} \subseteq \{x \in \Omega \mid \rho(x) < \mu^{-1}\}. \tag{2.11}$$

By Taylor's theorem, ϕ_μ can be written as

$$\phi_\mu(x) = \rho(x) - \mu[\rho(x)]^2 + \tfrac{1}{2\mu}|x - x_0|^2 + E(x),$$

where the error term $E(x)$ satisfies

$$|E(x)| \lesssim |x - x_0|^3, \qquad x \in U.$$

Then, by further shrinking ε if needed (depending on ρ and μ), we see that

$$\phi_\mu(x) \geq \rho(x)\,[1 - \mu\,\rho(x)] + \tfrac{1}{4\mu}|x - x_0|^2, \qquad x \in V. \tag{2.12}$$

In particular, (2.11) and (2.12) imply that $\phi_\mu(x) > 0$ for any $x \in V$ with $\rho(x) > 0$.

Next, fix a constant $0 < \delta < (4\mu)^{-1}\varepsilon^2$. In order to show (2.9), with our chosen V and δ, it suffices to prove that if $x \in \partial V$ and $\phi_\mu(x) \leq \delta$, then $\rho(x) < 0$. To establish this, we observe that if $x \in \partial V$ and $\phi_\mu(x) \leq \delta$, then (2.12) yields the inequalities

$$\rho(x)\,[1 - \mu\,\rho(x)] + \tfrac{1}{4\mu}\varepsilon^2 \leq \phi_\mu(x) \leq \delta, \qquad \rho(x)\,[1 - \mu\,\rho(x)] < 0.$$

Now, since $1 - \mu\,\rho(x) > 0$ by (2.11), it follows that $\rho(x) < 0$, as desired. □

We now state, as a hypothetical property, the general form of our Carleman estimate:

Definition 2.16. *We say that* $(\mathcal{P}, \Sigma, \rho)$ admits a Carleman estimate *at x_0 iff there exists a neighborhood $U \subset\subset \Omega$ of x_0, a Carleman weight $\phi \in C^\infty(\Omega; \mathbb{R})$ for ρ at x_0, and constants $C, \lambda_0 > 0$ such that for any $w \in C_0^\infty(U; \mathbb{C})$ and $\lambda \geq \lambda_0$,*

$$\lambda^{-\frac{1}{2}} \|e^{-\lambda\phi}\mathcal{P}_0 w\|_{L^2(U)} \geq C \sum_{j=0}^{m-1} \lambda^{m-1-j} \|e^{-\lambda\phi}\nabla^j w\|_{L^2(U)}. \tag{2.13}$$

The Carleman estimate property has an equivalent formulation, in which the exponential weight is removed at the cost of replacing $\mathcal{P}_0$ with a conjugated operator. In the remainder of this subsection, we describe this more convenient alternate formulation.

Definition 2.17. *Given any $\lambda > 0$, $s \in \mathbb{R}$, and open $U \subseteq \mathbb{R}^n$, we define the norm*

$$\|v\|_{\mathcal{H}^s_\lambda(U)} := \|(\lambda - \Delta)^{\frac{s}{2}} v\|_{L^2(\mathbb{R}^n)}, \qquad v \in C_0^\infty(U; \mathbb{C}). \tag{2.14}$$

Remark 2.18. *Note that powers of $\lambda - \Delta$ in* (2.14) *are well-defined, since v can be smoothly zero-extended to $\mathbb{R}^n$. Also, by repeated integrations by parts, we have that*

$$\|v\|_{\mathcal{H}^k_\lambda(U)} \simeq \sum_{j=0}^{k} \lambda^{k-j} \|\nabla^j v\|_{L^2(U)}, \qquad v \in C_0^\infty(U; \mathbb{C}) \tag{2.15}$$

for any $k \in \mathbb{N}$, where the constants depend only on k.

Proposition 2.19. $(\mathcal{P}, \Sigma, \rho)$ *admits a Carleman estimate at x_0 if and only if there exists a neighborhood $U \subset\subset \Omega$ of x_0, a Carleman weight $\phi \in C^\infty(\Omega; \mathbb{R})$ for ρ at x_0, and constants $C, \lambda_0 > 0$ such that for any $v \in C_0^\infty(U; \mathbb{C})$ and $\lambda \geq \lambda_0$,*

$$\|e^{-\lambda\phi}\mathcal{P}_0(e^{\lambda\phi} v)\|_{L^2(U)} \geq C\lambda^{\frac{1}{2}} \|v\|_{\mathcal{H}^{m-1}_\lambda(U)}. \tag{2.16}$$

Proof. Suppose first that (2.16) holds. Fix $w \in C_0^\infty(U; \mathbb{C})$, and let

$$v := e^{-\lambda\phi} w.$$

Then, since $\bar{U}$ is compact, we obtain

$$\begin{aligned}\sum_{j=0}^{m-1} \lambda^{m-1-j} \| e^{-\lambda\phi} \nabla^j w \|_{L^2(U)} &= \sum_{j=0}^{m-1} \lambda^{m-1-j} \| (e^{-\lambda\phi} \nabla e^{\lambda\phi})^j v \|_{L^2(U)} \\ &\lesssim \| v \|_{\mathcal{H}_\lambda^{m-1}(U)}.\end{aligned}$$

Applying (2.16), we then conclude, for $\lambda \geq \lambda_0$, that (2.13) holds:

$$\begin{aligned}\sum_{j=0}^{m-1} \lambda^{m-1-j} \| e^{-\lambda\phi} \nabla^j w \|_{L^2(U)} &\lesssim \lambda^{-\frac{1}{2}} \| e^{-\lambda\phi} \mathcal{P}_0 (e^{\lambda\phi} v) \|_{L^2(U)} \\ &= \lambda^{-\frac{1}{2}} \| e^{-\lambda\phi} \mathcal{P}_0 w \|_{L^2(U)}.\end{aligned}$$

Conversely, suppose that (2.13) holds. We now fix $v \in C_0^\infty(U; \mathbb{C})$, and we let

$$w := e^{\lambda\phi} v.$$

Then, recalling (2.15), we see that

$$\begin{aligned}\lambda^{\frac{1}{2}} \| v \|_{\mathcal{H}_\lambda^{m-1}(U)} &= \lambda^{\frac{1}{2}} \sum_{j=0}^{m-1} \lambda^{m-1-j} \| e^{-\lambda\phi} (e^{\lambda\phi} \nabla e^{-\lambda\phi})^j w \|_{L^2(U)} \\ &\lesssim \lambda^{\frac{1}{2}} \sum_{j=0}^{m-1} \lambda^{m-1-j} \| e^{-\lambda\phi} \nabla^j w \|_{L^2(U)}.\end{aligned}$$

Applying (2.13) and the above, we conclude, for any $\lambda \geq \lambda_0$, that

$$\begin{aligned}\lambda^{\frac{1}{2}} \| v \|_{\mathcal{H}_\lambda^{m-1}(U)} &\lesssim \| e^{-\lambda\phi} \mathcal{P}_0 w \|_{L^2(U)} \\ &\lesssim \| e^{-\lambda\phi} \mathcal{P}_0 (e^{\lambda\phi} v) \|_{L^2(U)},\end{aligned}$$

which completes the proof of (2.16). □

2.3. **Local Unique Continuation.** The next definition captures the precise formulation of the unique continuation problem that we wish to solve in these notes:

Definition 2.20. *We say that $(\mathcal{P}, \Sigma, \rho)$ has the* local unique continuation property *at x_0 iff there exists a neighborhood U of x_0 such that if $w \in C^\infty(\Omega; \mathbb{C})$ satisfies $\mathcal{P} w \equiv 0$ on Ω, and if $w \equiv 0$ on the region $\{x \in \Omega \mid \rho(x) \leq 0\}$, then $w \equiv 0$ on U as well.*

In other words, if $(\mathcal{P}, \Sigma, \rho)$ has the local unique continuation property at some $x_0 \in \Sigma$, then any solution of $\mathcal{P} w = 0$ that vanishes when $\rho \leq 0$ can be uniquely continued as a zero solution past Σ, at least near x_0. The main goal of the present notes is to identify conditions under which $(\mathcal{P}, \Sigma, \rho)$ has the unique continuation property at x_0.

Remark 2.21. *Definition 2.20 is sensitive to the chosen orientation of Σ. In particular, it is important that we are continuing solutions of $\mathcal{P} w = 0$ from $\rho \leq 0$ into $\rho > 0$.*

We now establish the connection between unique continuation and Carleman estimates. In short, proving a Carleman estimate leads to a corresponding unique continuation property:

Theorem 2.22. *Suppose that $(\mathcal{P}, \Sigma, \rho)$ admits a Carleman estimate at x_0. Then, $(\mathcal{P}, \Sigma, \rho)$ has the local unique continuation property at x_0.*

Proof. Let U, ϕ, and λ_0 be the objects from Definition 2.16 corresponding to the assumed Carleman estimate at x_0. Then, by Definition 2.12, we can find an open $V \subseteq U$ and $\delta > 0$ such that $\phi(x) > \phi(x_0)$ for any $x \in V$ with $\rho(x) > 0$, and such that (2.9) holds.

Furthermore, let $\chi \in C^\infty(\mathbb{R}; [0,1])$ be a cutoff function satisfying

$$\chi|_{[-\frac{\delta}{3}, \frac{\delta}{3}]} \equiv 1, \qquad \chi|_{\mathbb{R} \setminus [-\frac{2\delta}{3}, \frac{2\delta}{3}]} \equiv 0, \tag{2.17}$$

and define the following (non-empty by assumption) subsets of V:

$$\begin{aligned} V_< &:= \left\{ x \in V \,\middle|\, \rho(x) > 0,\, \phi(x) - \phi(x_0) \leq \tfrac{\delta}{3} \right\}, \\ V_> &:= \left\{ x \in V \,\middle|\, \rho(x) > 0,\, \tfrac{\delta}{3} \leq \phi(x) - \phi(x_0) \leq \tfrac{2\delta}{3} \right\}. \end{aligned} \tag{2.18}$$

Now, let $w \in C^\infty(\Omega; \mathbb{C})$ satisfy $\mathcal{P} w \equiv 0$, and assume $w \equiv 0$ on $\{\rho \leq 0\}$. By (2.9),

$$\operatorname{supp}[\chi(\phi)\, w] \subseteq \{ x \in \bar{V} \mid \rho(x) \geq 0,\, \phi(x) - \phi(x_0) \leq \delta \} \subseteq V,$$

and hence $\chi(\phi)\, w \in C^\infty_0(U; \mathbb{C})$. Consequently, we can apply Definition 2.16 to $\chi(\phi)\, w$, which yields the following Carleman estimate for each $\lambda \geq \lambda_0$:

$$\begin{aligned} \sum_{j=0}^{m-1} \lambda^{m-1-j} \| e^{-\lambda\phi} \nabla^j w \|_{L^2(V_<)} &\leq \sum_{j=0}^{m-1} \lambda^{m-1-j} \| e^{-\lambda\phi} \nabla^j [\chi(\phi)\, w] \|_{L^2(U)} \\ &\lesssim \lambda^{-\frac{1}{2}} \| e^{-\lambda\phi} \mathcal{P}_0 [\chi(\phi)\, w] \|_{L^2(U)}. \end{aligned} \tag{2.19}$$

Recalling that $\mathcal{P} w \equiv 0$, we can then expand and estimate

$$\begin{aligned} |\mathcal{P}_0 [\chi(\phi)\, w]| &\lesssim \chi(\phi)\, |\mathcal{P}_0 w| + \sum_{j=0}^{m-1} \sum_{k=1}^{m-j} |\chi^{(k)}(\phi)| |\nabla^j w| \\ &\lesssim \sum_{j=0}^{m-1} \sum_{k=0}^{m-j} |\chi^{(k)}(\phi)| |\nabla^j w|, \end{aligned} \tag{2.20}$$

where the constants of the inequalities can depend on ϕ and the coefficients of $\mathcal{P}$ (but not on w). Moreover, from (2.17) and (2.18), we see that the right-hand side of (2.20) is supported in $V_< \cup V_>$. Thus, combining (2.19) and (2.20), we obtain the estimate

$$\sum_{j=0}^{m-1} \lambda^{m-1-j} \| e^{-\lambda\phi} \nabla^j w \|_{L^2(V_<)} \lesssim \lambda^{-\frac{1}{2}} \sum_{j=0}^{m-1} \| e^{-\lambda\phi} \nabla^j w \|_{L^2(V_<)} + \lambda^{-\frac{1}{2}} \sum_{j=0}^{m-1} \| e^{-\lambda\phi} \nabla^j w \|_{L^2(V_>)}.$$

Also, for large enough λ, the first term on the right can be absorbed into the left:

$$\sum_{j=0}^{m-1} \| e^{-\lambda\phi} \nabla^j w \|_{L^2(V_<)} \lesssim \lambda^{-\frac{1}{2}} \sum_{j=0}^{m-1} \| e^{-\lambda\phi} \nabla^j w \|_{L^2(V_>)}. \tag{2.21}$$

From the definitions (2.18), we see that

$$e^{-\lambda\phi}|_{V_<} \geq e^{-\frac{\lambda\delta}{3}}, \qquad e^{-\lambda\phi}|_{V_>} \leq e^{-\frac{\lambda\delta}{3}}.$$

Using the above, we can then remove the exponential weights from (2.21):

$$\sum_{j=0}^{m-1} \|\nabla^j w\|_{L^2(V_<)} \lesssim \lambda^{-\frac{1}{2}} \sum_{j=0}^{m-1} \|\nabla^j w\|_{L^2(V_>)}.$$

Taking $\lambda \nearrow \infty$, we conclude that $w \equiv 0$ on $V_<$.

From the above, along with our original assumptions for w, it follows that $w \equiv 0$ on the neighborhood $W := V \cap \{\phi - \phi(x_0) < \frac{\delta}{3}\}$ of x_0. Finally, as W is independent of w, we have shown that $(\mathcal{P}, \Sigma, \rho)$ satisfies the local unique continuation property at x_0. $\square$

In particular, Theorem 2.22 reduces the problem of deriving the local unique continuation property to that of proving a Carleman estimate. Thus, in the remainder of these notes, we will focus our attention on establishing Carleman estimates.

3. The Symbol Calculus

The aim of these notes is to study Carleman estimates through microlocal methods. In this section, we give a brief survey of the microlocal background that we will need to prove our Carleman estimates. Then, we reformulate the conjugated Carleman estimate (2.16) as a corresponding pointwise inequality for the appropriate symbols.

3.1. λ-Parametrized Symbols.

A key idea in microlocal analysis is to systematically study classes of operators through corresponding classes of functions known as symbols. While we wish to do this here for Carleman estimates, the process runs into some complications.

In particular, from Proposition 2.19, we saw that in order to prove Carleman estimates for $\mathcal{P}_0$, we must deal with conjugated operators of the form $e^{-\lambda\phi}\mathcal{P}_0 e^{\lambda\phi}$, which contain terms with lower-order derivatives paired with an arbitrarily large parameter λ. Thus, we must also view these lower-order terms as "principal" in our analysis. This motivates the construction of a modified "λ-dependent" symbol calculus, which we describe below:

Definition 3.1. *Given $s \in \mathbb{R}$, we let $\Lambda^s(\mathbb{R}^n)$ be the space of all $b \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times (0,\infty); \mathbb{C})$ such that for any multi-indices I, J, there exists $C > 0$ (depending b, I, J) such that*

$$|\nabla_{x,I} \nabla_{\xi,J} b(x,\xi,\lambda)| \leq C(\lambda^2 + |\xi|^2)^{\frac{s-|J|}{2}}, \qquad (x,\xi,\lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times [1,\infty). \tag{3.1}$$

Such a function b is called a (smooth, λ-parametrized) symbol *on $\mathbb{R}^n$ of* order *s.*

We can then associate such symbols with a corresponding class of operators.

Definition 3.2. *We will adopt the following convention for the Fourier transform:*

$$\hat{\varphi}(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \varphi(x)\, dx, \qquad \varphi \in C_0^\infty(\mathbb{R}^n; \mathbb{C}). \tag{3.2}$$

Definition 3.3. *Given $s \in \mathbb{R}$, $\lambda \in [1,\infty)$, and $b \in \Lambda^s(\mathbb{R}^n)$, we define the operator $\mathrm{op}_\lambda(b)$ by*

$$\mathrm{op}_\lambda(b)\psi(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} b(x,\xi,\lambda) \hat{\psi}(\xi)\, d\xi, \qquad \psi \in C_0^\infty(\mathbb{R}^n; \mathbb{C}), \quad x \in \mathbb{R}^n. \tag{3.3}$$

$\mathrm{op}_\lambda(b)$ is called a (λ-parametrized) pseudodifferential operator *on $\mathbb{R}^n$ of* order *s.*

Remark 3.4. *For any integer $M \geq 0$, the (λ-parametrized) polynomial*

$$b \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times (0,\infty); \mathbb{C}), \qquad b(x,\xi,\lambda) = \sum_{|I|+j \leq M} b^{I,j}(x)\, \xi_I \lambda^j, \tag{3.4}$$

with $b^{I,j} \in C_0^\infty(\mathbb{R}^n; \mathbb{C})$ for each such I and j, defines an element of $\Lambda^M(\mathbb{R}^n)$. By elementary properties of Fourier transforms, b is associated with a family of differential operators:

$$\mathrm{op}_\lambda(b) = \sum_{|I|+j \leq M} b^{I,j} \lambda^j\, D_I, \qquad \lambda \in [1,\infty). \tag{3.5}$$

Remark 3.5. *Observe also that for any $s \in \mathbb{R}$, the function*

$$\gamma^s \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times (0,\infty); \mathbb{C}), \qquad \gamma^s(x,\xi,\lambda) := (\lambda^2 + |\xi|)^{\frac{s}{2}} \tag{3.6}$$

defines an element of $\Lambda^s(\mathbb{R}^n)$*. Furthermore, a direct computation using* (3.3) *yields*

$$\mathrm{op}_\lambda(\gamma^s) = (\lambda^2 - \Delta)^{\frac{s}{2}}, \qquad \lambda \in [1, \infty). \tag{3.7}$$

Proposition 3.6. *Let* $s \in \mathbb{R}$ *and* $b \in \Lambda^s(\mathbb{R}^n)$*. Then:*

- *There exists* $C > 0$ *such that for any* $\psi \in C_0^\infty(\mathbb{R}^n; \mathbb{C})$ *and* $\lambda \in [1, \infty)$*,*

$$\| \mathrm{op}_\lambda(b)\psi\|_{L^2(\mathbb{R}^n)} \leq C\|\psi\|_{\mathcal{H}^s_\lambda(\mathbb{R}^n)}. \tag{3.8}$$

- *There exists* $C > 0$ *such that for any* $\psi \in C_0^\infty(\mathbb{R}^n; \mathbb{C})$ *and* $\lambda \in [1, \infty)$*,*

$$\langle \mathrm{op}_\lambda(b)\psi, \psi\rangle_{L^2(\mathbb{R}^n)} \leq C\|\psi\|_{\mathcal{H}^{\frac{s}{2}}_\lambda(\mathbb{R}^n)}. \tag{3.9}$$

Proof. (We defer the proof to the Appendix to avoid an extended technical detour.) □

One quantity that will often arise is the Poisson bracket, which we now recall:

Definition 3.7. *Given* $q_1, q_2 \in C^\infty(U \times \mathbb{R}^n; \mathbb{C})$ *and* $b_1, b_2 \in C^\infty(U \times \mathbb{R}^n \times (0, \infty); \mathbb{C})$*, where* $U \subseteq \mathbb{R}^n$ *is open, we define their* Poisson bracket *by*

$$\begin{aligned} \{q_1, q_2\} &:= \nabla_\xi q_1 \cdot \nabla_x q_2 - \nabla_x q_1 \cdot \nabla_\xi q_2 \in C^\infty(U \times \mathbb{R}^n; \mathbb{C}), \\ \{b_1, b_2\} &:= \nabla_\xi b_1 \cdot \nabla_x b_2 - \nabla_x b_1 \cdot \nabla_\xi b_2 \in C^\infty(U \times \mathbb{R}^n \times (0, \infty); \mathbb{C}), \end{aligned} \tag{3.10}$$

where ∇_x *and* ∇_ξ *denote gradients in the "*U*" and "*$\mathbb{R}^n$*" components, respectively.*

Remark 3.8. *In the context of Definition 3.7, any* $\psi \in C^\infty(U; \mathbb{C})$ *can also be viewed as an element of* $C^\infty(U \times \mathbb{R}^n; \mathbb{C})$ *that is independent of the latter argument. This allows one to make sense of Poisson brackets involving* ψ*—e.g., for any* $q \in C^\infty(U \times \mathbb{R}^n; \mathbb{C})$*, we have*

$$\{q, \psi\}(x, \xi) = \nabla_\xi q(x, \xi) \cdot \nabla\psi(x), \qquad (x, \xi) \in U \times \mathbb{R}^n.$$

Remark 3.9. *Note that if* $b_1 \in \Lambda^{s_1}(\mathbb{R}^n)$ *and* $b_2 \in \Lambda^{s_2}(\mathbb{R}^n)$*, with* $s_1, s_2 \in \mathbb{R}$*, then*

$$\{b_1, b_2\} \in \Lambda^{s_1+s_2-1}(\mathbb{R}^n).$$

We now recall some standard correspondences between algebraic operations on pseudodifferential operators and algebraic operations on the corresponding symbols:

Proposition 3.10. *Let* $s, s_1, s_2 \in \mathbb{R}$*, and let* $b \in \Lambda^s(\mathbb{R}^n)$*,* $b_1 \in \Lambda^{s_1}(\mathbb{R}^n)$*,* $b_2 \in \Lambda^{s_2}(\mathbb{R}^n)$*:*

- *There exists* $r \in \Lambda^{s_1+s_2-2}(\mathbb{R}^n)$ *such that*

$$\mathrm{op}_\lambda(b_1)\,\mathrm{op}_\lambda(b_2) = \mathrm{op}_\lambda(b_1 b_2 - i\nabla_\xi b_1 \cdot \nabla_x b_2 + r), \qquad \lambda \in [1, \infty). \tag{3.11}$$

- *There exists* $r \in \Lambda^{s_1+s_2-2}(\mathbb{R}^n)$ *such that*

$$[\mathrm{op}_\lambda(b_1), \mathrm{op}_\lambda(b_2)] = \mathrm{op}_\lambda(-i\{b_1, b_2\} + r), \qquad \lambda \in [1, \infty). \tag{3.12}$$

- *There exists* $r \in \Lambda^{s-1}(\mathbb{R}^n)$ *such that*

$$\mathrm{op}_\lambda(b)^* = \mathrm{op}_\lambda(\bar{b} + r), \qquad \lambda \in [1, \infty). \tag{3.13}$$

Proof. (We defer the proof to the Appendix to avoid an extended technical detour.) □

The final ingredient from microlocal analysis that we will need is a way to convert symbol inequalities to bounds for corresponding operators. This is achieved through a semiclassical variant of the sharp Gårding inequality, for which we give a precise statement below:

Theorem 3.11. *Let $s \in \mathbb{R}$ and $\lambda_0 \geq 1$, and suppose $b \in \Lambda^s(\mathbb{R}^n)$ satisfies*

$$b(x, \xi, \lambda) \geq 0, \qquad (x, \xi, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times [\lambda_0, \infty). \tag{3.14}$$

Then, there exists $C > 0$ such that for any $\lambda \geq \lambda_0$ and $\psi \in C_0^\infty(\mathbb{R}^n; \mathbb{C})$,

$$\operatorname{Re}\langle \operatorname{op}_\lambda(b)\psi, \psi \rangle_{L^2(\mathbb{R}^n)} \geq -C \|\psi\|^2_{\mathcal{H}_\lambda^{\frac{s-1}{2}}(\mathbb{R}^n)}. \tag{3.15}$$

Proof. (We defer the proof to the Appendix to avoid an extended technical detour.) □

Remark 3.12. *The crucial semiclassical feature of* (3.15)*, in contrast to the basic sharp Gårding inequality, is that* (3.15)*—in particular the constant C—is uniform in λ.*

3.2. **Positive Commutators.** We now turn our attention back to our differential operator $\mathcal{P}$. First, we associate transform its principal symbol p to a conjugated symbol:

Definition 3.13. *Given any $\phi \in C^\infty(\Omega; \mathbb{R})$, we define $p_\phi \in C^\infty(\Omega \times \mathbb{R}^n \times (0, \infty); \mathbb{C})$ by*

$$p_\phi(x, \xi, \lambda) := p(x, \xi - i\lambda \, d\phi(x)). \tag{3.16}$$

Remark 3.14. *Note if $\chi \in C_0^\infty(\Omega; \mathbb{R})$ is a cutoff, then we can define an associated symbol*

$$\chi p_\phi \in \Lambda^m(\mathbb{R}^n), \qquad (\chi p_\phi)(x, \xi, \lambda) := \chi(x) \, p_\phi(x, \xi, \lambda).$$

Proposition 3.15. *Let $\phi \in C^\infty(\Omega; \mathbb{R})$, and suppose $U \subset\subset \Omega$ is open. Then, there is some $r_\phi \in \Lambda^{m-1}(\mathbb{R}^n)$ such that for any cutoff $\chi \in C^\infty(\Omega; \mathbb{R})$ satisfying $\chi|_U \equiv 1$, we have*

$$e^{-\lambda\phi} \mathcal{P}_0(e^{\lambda\phi} v) = \operatorname{op}_\lambda(\chi p_\phi + r_\phi) v, \qquad \lambda \in [1, \infty), \quad v \in C_0^\infty(U; \mathbb{C}). \tag{3.17}$$

Proof. First, observe that the left-hand side of (3.17) can be written as

$$e^{-\lambda\phi} \mathcal{P}_0 e^{\lambda\phi} = \sum_{|I|=m} p^I \, (D - i\lambda \, d\phi)_I. \tag{3.18}$$

Moreover, by (2.6), (3.3), and (3.16),

$$\operatorname{op}_\lambda(\chi p_\phi) v(x) = \frac{\chi(x)}{(2\pi)^n} \sum_{|I|=m} p^I(x) \int_{\mathbb{R}^n} e^{ix\cdot\xi} (\xi - i\lambda d\phi(x))_I \hat{v}(\xi) \, d\xi,$$

which, by (3.18) and basic properties of the Fourier transform, coincides with $\chi e^{-\lambda\phi} \mathcal{P}_0(e^{\lambda\phi} v)$, except for terms obtained when instances of D in the right-hand side of (3.18) hit instances of $d\phi$. These terms together comprise a differential operator of the form

$$\mathcal{R} := \sum_{|I|+j<m} r^I \, \lambda^j D_I, \qquad r^I \in C_0^\infty(\mathbb{R}^n; \mathbb{C}), \quad \lambda \in [1, \infty).$$

As a result, we obtain that

$$\chi e^{-\lambda\phi} \mathcal{P}_0 e^{\lambda\phi} - \operatorname{op}_\lambda(\chi p_\phi) = \operatorname{op}_\lambda(r_\phi), \qquad r_\phi := \sum_{|I|+j<m} r^I \, \lambda^j \xi_I \in \Lambda^{m-1}(\mathbb{R}^n).$$

The desired (3.17) now follows, since $\chi \equiv 1$ on the support of any $v \in C_0^\infty(U;\mathbb{C})$. □

Finally, we reduce the Carleman estimate (2.13) to a commutator estimate for p_ϕ:

Proposition 3.16. *$(\mathcal{P}, \Sigma, \rho)$ admits a Carleman estimate at x_0 if there exist a neighborhood $U' \subset\subset \Omega$ of x_0, a Carleman weight $\phi \in C^\infty(\Omega;\mathbb{R})$ for ρ at x_0, and constants $C > 0$, $\lambda_0 \geq 1$ such that the following inequality holds for every $x \in U'$, $\xi \in \mathbb{R}^n$, and $\lambda \geq \lambda_0$:*

$$\lambda_0(\lambda^2 + |\xi|^2)^{-\frac{1}{2}} |p_\phi(x,\xi,\lambda)|^2 + \{\operatorname{Re} p_\phi, \operatorname{Im} p_\phi\}(x,\xi,\lambda) \geq C\lambda(\lambda^2 + |\xi|^2)^{m-1}. \tag{3.19}$$

Proof. Fix a neighborhood $U \subset\subset U'$ of x_0, as well as a cutoff function

$$\chi \in C_0^\infty(U';[0,1]), \qquad \chi|_U \equiv 1. \tag{3.20}$$

Moreover, let C be as in (3.19), and let $b \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times (0,\infty);\mathbb{C})$ be given by

$$\begin{aligned} b(x,\xi,\lambda) &:= [b_r(x,\xi,\lambda)]^2 + [b_i(x,\xi,\lambda)]^2 + b_c(x,\xi,\lambda) + b_-(x,\xi,\lambda), \\ b_r(x,\xi,\lambda) &:= \chi(x)\, \lambda_0^{\frac{1}{2}} (\lambda^2 + |\xi|^2)^{-\frac{1}{4}} \operatorname{Re} p_\phi(x,\xi,\lambda), \\ b_i(x,\xi,\lambda) &:= \chi(x)\, \lambda_0^{\frac{1}{2}} (\lambda^2 + |\xi|^2)^{-\frac{1}{4}} \operatorname{Im} p_\phi(x,\xi,\lambda), \\ b_c(x,\xi,\lambda) &:= [\chi(x)]^2 \{\operatorname{Re} p_\phi, \operatorname{Im} p_\phi\}(x,\xi,\lambda), \\ b_-(x,\xi,\lambda) &:= -[\chi(x)]^2\, C\lambda(\lambda^2 + |\xi|^2)^{m-1}. \end{aligned} \tag{3.21}$$

Note in particular that $b \in \Lambda^{2m-1}(\mathbb{R}^n)$, and that (3.19) and (3.20) imply

$$b(x,\xi,\lambda) \geq 0, \qquad (x,\xi,\lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times [\lambda_0, \infty).$$

Thus, Theorem 3.11 yields some $C' > 0$ such that for any $v \in C_0^\infty(U;\mathbb{C})$ and $\lambda \geq \lambda_0$,

$$\begin{aligned} -C' \|v\|^2_{\mathcal{H}^{m-1}_\lambda(U)} &\leq \operatorname{Re}\langle \operatorname{op}_\lambda(b_r^2) v, v\rangle_{L^2(\mathbb{R}^n)} + \operatorname{Re}\langle \operatorname{op}_\lambda(b_i^2) v, v\rangle_{L^2(\mathbb{R}^n)} \\ &\qquad + \operatorname{Re}\langle \operatorname{op}_\lambda(b_c) v, v\rangle_{L^2(\mathbb{R}^n)} + \operatorname{Re}\langle \operatorname{op}_\lambda(b_-) v, v\rangle_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{3.22}$$

In (3.22) and below, we will let $C', C'' > 0$ denote various constants which are independent of v and λ, and whose values are allowed to change between lines. Recalling (3.3), and noting that v is supported only where χ is identically 1, we obtain

$$\begin{aligned} \langle \operatorname{op}_\lambda(b_-) v, v\rangle_{L^2(\mathbb{R}^n)} &= -\frac{C}{(2\pi)^n} \int_{\mathbb{R}^n} \lambda(\lambda^2 + |\xi|^2)^{m-1} \hat{v}(\xi) \Big[\int_{\mathbb{R}^n} \overline{e^{-ix\cdot\xi} v(x)}\, dx \Big] d\xi \\ &\leq -C''\lambda \|v\|^2_{\mathcal{H}^{m-1}_\lambda(U)}, \end{aligned} \tag{3.23}$$

for v and λ as above. Furthermore, since $\operatorname{op}_\lambda(b_c)$ is a differential (and hence local) operator, and hence $\chi \equiv 1$ on the support of v, then (3.3) also yields that

$$\begin{aligned} \langle \operatorname{op}_\lambda(b_c) v, v\rangle_{L^2(\mathbb{R}^n)} &= \frac{1}{(2\pi)^n} \int_U \int_{\mathbb{R}^n} e^{ix\cdot\xi} \{\operatorname{Re} p_\phi, \operatorname{Im} p_\phi\}(x,\xi,\lambda) \hat{v}(\xi) \overline{v(x)}\, d\xi dx \\ &= \langle \operatorname{op}_\lambda(\{\chi \operatorname{Re} p_\phi, \chi \operatorname{Im} p_\phi\}) v, v\rangle_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{3.24}$$

Combining (3.22)–(3.24) then yields

$$C''\lambda \|v\|^2_{\mathcal{H}^{m-1}_\lambda(U)} \leq \operatorname{Re}\langle \operatorname{op}_\lambda(b_r^2) v, v\rangle_{L^2(\mathbb{R}^n)} + \operatorname{Re}\langle \operatorname{op}_\lambda(b_i^2) v, v\rangle_{L^2(\mathbb{R}^n)} \tag{3.25}$$

$$+ \operatorname{Re}\langle \operatorname{op}_\lambda(\{\chi \operatorname{Re} p_\phi, \chi \operatorname{Im} p_\phi\}) v, v\rangle_{L^2(\mathbb{R}^n)} + C' \|v\|^2_{\mathcal{H}^{m-1}_\lambda(U)}.$$

Noting that $b_r, b_i \in \Lambda^{m-\frac{1}{2}}(\mathbb{R}^n)$ are both real valued, then Proposition 3.10 implies

$$\operatorname{op}_\lambda(b_r)^* \operatorname{op}_\lambda(b_r) = \operatorname{op}_\lambda(b_r^2 + \rho_r), \qquad \operatorname{op}_\lambda(b_i)^* \operatorname{op}_\lambda(b_i) = \operatorname{op}_\lambda(b_i^2 + \rho_i),$$

for some $\rho_r, \rho_i \in \Lambda^{2m-2}(\mathbb{R}^n)$. In particular, Proposition 3.6 and the above yield

$$\begin{aligned} \langle \operatorname{op}_\lambda(b_r^2) v, v\rangle_{L^2(\mathbb{R}^n)} &\leq \|\operatorname{op}_\lambda(b_r) v\|^2_{L^2(\mathbb{R}^n)} + C' \|v\|^2_{\mathcal{H}^{m-1}_\lambda(U)}, \\ \langle \operatorname{op}_\lambda(b_i^2) v, v\rangle_{L^2(\mathbb{R}^n)} &\leq \|\operatorname{op}_\lambda(b_i) v\|^2_{L^2(\mathbb{R}^n)} + C' \|v\|^2_{\mathcal{H}^{m-1}_\lambda(U)}. \end{aligned} \tag{3.26}$$

Next, we estimate the L^2-norms of $\operatorname{op}_\lambda(b_r)v$ and $\operatorname{op}_\lambda(b_i)v$ using (3.7), Proposition 3.6, Proposition 3.10, and the definitions (3.21) of b_r and b_i. Combining this with (3.26) yields

$$\begin{aligned} \langle \operatorname{op}_\lambda(b_r^2) v, v\rangle_{L^2(\mathbb{R}^n)} &\leq \left\|\lambda_0^{\frac{1}{2}} (\lambda^2 - \Delta)^{-\frac{1}{4}} \operatorname{op}_\lambda(\chi \operatorname{Re} p_\phi) v\right\|^2_{L^2(\mathbb{R}^n)} + C' \|v\|^2_{\mathcal{H}^{m-1}_\lambda(U)}, \\ \langle \operatorname{op}_\lambda(b_i^2) v, v\rangle_{L^2(\mathbb{R}^n)} &\leq \left\|\lambda_0^{\frac{1}{2}} (\lambda^2 - \Delta)^{-\frac{1}{4}} \operatorname{op}_\lambda(\chi \operatorname{Im} p_\phi) v\right\|^2_{L^2(\mathbb{R}^n)} + C' \|v\|^2_{\mathcal{H}^{m-1}_\lambda(U)}, \end{aligned} \tag{3.27}$$

Next, consider the operators

$$J := \tfrac{1}{2}[\operatorname{op}_\lambda(\chi p_\phi) + \operatorname{op}_\lambda(\chi p_\phi)^*], \qquad K := \tfrac{1}{2}[\operatorname{op}_\lambda(\chi p_\phi) - \operatorname{op}_\lambda(\chi p_\phi)^*]. \tag{3.28}$$

Applying Proposition 3.10 to the above, we can then write

$$\begin{aligned} J &= \operatorname{op}_\lambda(\chi \operatorname{Re} p_\phi + \alpha_r), \qquad && \alpha_r \in \Lambda^{m-1}(\mathbb{R}^n), \\ K &= i \operatorname{op}_\lambda(\chi \operatorname{Im} p_\phi + \alpha_i), && \alpha_i \in \Lambda^{m-1}(\mathbb{R}^n), \\ [J, K] &= \operatorname{op}_\lambda(\{\chi \operatorname{Re} p_\phi, \chi \operatorname{Im} p_\phi\} + \alpha_c), && \alpha_c \in \Lambda^{2m-2}(\mathbb{R}^n). \end{aligned}$$

Applying the above along with Proposition 3.6, (3.25), and (3.27), we then obtain

$$\begin{aligned} C'' \lambda \|v\|^2_{\mathcal{H}^{m-1}_\lambda(U)} &\leq \left\|\lambda_0^{\frac{1}{2}} (\lambda^2 - \Delta)^{-\frac{1}{4}} J v\right\|^2_{L^2(\mathbb{R}^n)} + \left\|\lambda_0^{\frac{1}{2}} (\lambda^2 - \Delta)^{-\frac{1}{4}} K v\right\|^2_{L^2(\mathbb{R}^n)} \\ &\qquad + \operatorname{Re}\langle [J, K] v, v\rangle_{L^2(\mathbb{R}^n)} + C' \|v\|^2_{\mathcal{H}^{m-1}_\lambda(U)} \\ &\leq \|J v\|^2_{L^2(\mathbb{R}^n)} + \|K v\|^2_{L^2(\mathbb{R}^n)} + \operatorname{Re}\langle [J, K] v, v\rangle_{L^2(\mathbb{R}^n)} + C' \|v\|^2_{\mathcal{H}^{m-1}_\lambda(U)}, \end{aligned} \tag{3.29}$$

where in the last step, we also used that $\lambda \geq \lambda_0$.

Since $J^* = J$ and $K^* = -K$, and since v has compact support, a computation yields

$$\begin{aligned} \|\operatorname{op}_\lambda(\chi p_\phi) v\|^2_{L^2(\mathbb{R}^n)} &= \langle (J + K) v, (J + K) v\rangle_{L^2(\mathbb{R}^n)} \\ &= \|J v\|^2_{L^2(\mathbb{R}^n)} + \|K v\|^2_{L^2(\mathbb{R}^n)} + \operatorname{Re}\langle [J, K] v, v\rangle_{L^2(\mathbb{R}^n)}. \end{aligned}$$

The above, combined with (3.29), results in the bound

$$C'' \lambda \|v\|^2_{\mathcal{H}^{m-1}_\lambda(U)} \leq \|\operatorname{op}_\lambda(\chi p_\phi) v\|^2_{L^2(\mathbb{R}^n)} + C' \|v\|^2_{\mathcal{H}^{m-1}_\lambda(U)}.$$

Finally, applying Proposition 3.6 and Proposition 3.15 to the above yields

$$C'' \lambda \|v\|^2_{\mathcal{H}^{m-1}_\lambda(U)} \leq \|e^{-\lambda \phi} \mathcal{P}_0 (e^{\lambda \phi} v)\|^2_{L^2(U)} + C' \|v\|^2_{\mathcal{H}^{m-1}_\lambda(U)}.$$

By increasing λ_0 if needed, the last term on the right-hand side can be absorbed into the left, and we obtain the estimate (2.16); the result now follows from Proposition 2.19. □

4. Pseudoconvexity and Unique Continuation

In this section, we state and prove Hörmander's general local unique continuation result for linear differential operators, for which the crucial condition needed is strong pseudoconvexity. Under this assumption, we prove a Carleman estimate of the form (2.13).

4.1. **Pseudoconvexity.**

The first task is give a precise definition of pseudoconvexity:

Definition 4.1. *(Σ, ρ) is* pseudoconvex *at x_0 with respect to $\mathcal{P}$ iff for all $\xi \in \mathbb{R}^n \setminus \{0\}$, if*

$$p(x_0, \xi) = \{p, \rho\}(x_0, \xi) = 0, \tag{4.1}$$

then the following inequality holds:

$$-\operatorname{Re}\{\bar{p}, \{p, \rho\}\}(x_0, \xi) > 0. \tag{4.2}$$

Definition 4.2. *(Σ, ρ) is* strongly pseudoconvex *at x_0 with respect to $\mathcal{P}$ iff:*

- *(Σ, ρ) is pseudoconvex at x_0 with respect to $\mathcal{P}$.*
- *For any $\lambda > 0$ and $\xi \in \mathbb{R}^n$, if*

$$p_\rho(x_0, \xi, \lambda) = \{p_\rho, \rho\}(x_0, \xi, \lambda) = 0, \tag{4.3}$$

then the following inequality holds:

$$\{\operatorname{Re} p_\rho, \operatorname{Im} p_\rho\}(x_0, \xi, \lambda) > 0. \tag{4.4}$$

Remark 4.3. *For future reference, we note here the identity*

$$\{\operatorname{Re} p_\phi, \operatorname{Im} p_\phi\} = \tfrac{1}{2i}\{\overline{p_\phi}, p_\phi\}, \qquad \phi \in C^\infty(\Omega; \mathbb{R}). \tag{4.5}$$

Remark 4.4. *Direct computations using* (3.16) *yield, for any $\xi \in \mathbb{R}^n$, that*

$$\lim_{\lambda \searrow 0} p_\rho(x_0, \xi, \lambda) = p(x_0, \xi), \qquad \lim_{\lambda \searrow 0} \{p_\rho, \rho\}(x_0, \xi, \lambda) = \{p, \rho\}(x_0, \xi). \tag{4.6}$$

Thus, the assumption (4.1) *(for $\xi \neq 0$) is simply the condition* (4.3) *as $\lambda \searrow 0$.*

Strong pseudoconvexity will be the crucial condition required for our most general local unique continuation result. However, there are simpler special cases worth mentioning:

Proposition 4.5. *Suppose the following conditions hold:*

- *Given any $\xi \in \mathbb{R}^n \setminus \{0\}$, if $p(x_0, \xi) = 0$, then $\{p, \rho\}(x_0, \xi) \neq 0$.*
- *For any $(\xi, \lambda) \in \mathbb{R}^n \times (0, \infty)$, if $p_\rho(x_0, \xi, \lambda) = 0$, then $\{p_\rho, \rho\}(x_0, \xi, \lambda) \neq 0$.*

Then, (Σ, ρ) is strongly pseudoconvex at x_0.

Proof. First, for any $\lambda > 0$ and $\xi \in \mathbb{R}^n$, our second assumption implies that (4.3) can never hold. Thus, the second condition in Definition 4.2 is vacuously satisfied.

Similarly, for any $\xi \in \mathbb{R}^n \setminus \{0\}$, our first assumption implies (4.1) can never hold. Thus, (Σ, ρ) is pseudoconvex at x_0 with respect to $\mathcal{P}$, which completes the proof. □

Proposition 4.6. *Suppose that Σ is strictly hyperbolic at x_0 with respect to $\mathcal{P}$. Then, (Σ, ρ) is also strongly pseudoconvex at x_0 with respect to $\mathcal{P}$.*

Proof. Fix $(\xi, \lambda) \in \mathbb{R}^n \times (0, \infty)$. First, if $\xi \wedge d\rho(x_0) \neq 0$, then

$$-i\{p_\rho, \rho\}(x_0, \xi, \lambda) = \partial_\tau [p(x_0, \xi - i\tau\, d\rho(x_0))]|_{\tau=\lambda} \,,$$
$$-i\{p, \rho\}(x_0, \xi) = \partial_\tau [p(x_0, \xi - i\tau\, d\rho(x_0))]|_{\tau=0} \,,$$

and both of the above are nonzero, since by Definition 2.11, the polynomial $f_{p,x_0,\xi}$ defined in (2.8) (with our current ξ) has only simple roots.

Next, suppose $\xi \wedge d\rho(x_0) = 0$. Then, there is some $\sigma \in \mathbb{C} \setminus \{0\}$ such that

$$\xi - i\lambda\, d\rho(x_0) = \sigma\, d\rho(x_0), \qquad p_\rho(x_0, \xi, \lambda) = \sigma^m\, p(x_0, d\rho(x_0)).$$

In particular, since Σ is non-characteristic at x_0, we have $p_\rho(x_0, \xi, \lambda) \neq 0$. Similarly, if $\xi \neq 0$, then there is again some $\sigma \in \mathbb{C} \setminus \{0\}$ satisfying that

$$\xi = \sigma\, d\rho(x_0), \qquad p(x_0, \xi) = \sigma^m\, p(x_0, d\rho(x_0)),$$

the latter of which is nonzero since Σ is non-characteristic at x_0.

Thus, the assumptions of Proposition 4.5 are now satisfied, and the result follows. $\square$

We conclude with some explicit formulas for the quantities in (4.2) and (4.4):

Proposition 4.7. *The following hold for any $\phi \in C^\infty(\Omega; \mathbb{R})$, $\xi \in \mathbb{R}^n$, and $\lambda \in (0, \infty)$:*

$$\begin{aligned} -\operatorname{Re}\{\bar{p}, \{p, \phi\}\}(x_0, \xi) &= -\nabla_\xi \bar{p}(x_0, \xi) \cdot \nabla^2 \phi(x_0) \cdot \nabla_\xi p(x_0, \xi) \\ &\quad - \operatorname{Re} \sum_{\alpha,\beta=1}^n \partial^2_{x^\alpha \xi_\beta} p(x_0, \xi) \partial_{\xi_\alpha} \bar{p}(x_0, \xi) \partial_{x^\beta} \phi(x_0) \\ &\quad + \operatorname{Re} \sum_{\alpha,\beta=1}^n \partial^2_{\xi_\alpha \xi_\beta} p(x_0, \xi) \partial_{x^\alpha} \bar{p}(x_0, \xi) \partial_{x^\beta} \phi(x_0), \\ -i\{\overline{p_\phi}, p_\phi\}(x_0, \xi, \lambda) &= \operatorname{Im}[\nabla_\xi \bar{p}(x, \xi - i\lambda\, d\phi(x)) \cdot \nabla_x p(x, \xi - i\lambda\, d\phi(x))] \\ &\quad - \lambda\, \nabla_\xi \bar{p}(x, \xi - i\lambda\, d\phi(x)) \cdot \nabla^2 \phi(x) \cdot \nabla_\xi p(x, \xi - i\lambda\, d\phi(x)). \end{aligned} \tag{4.7}$$

Proof. For the first part of (4.7), we begin by computing

$$\begin{aligned} \{\bar{p}, \{p, \phi\}\} &= \{\bar{p}, \nabla_\xi p \cdot \nabla \phi\} \\ &= \nabla_\xi \bar{p} \cdot \nabla_x (\nabla_\xi p \cdot \nabla \phi) - \nabla_x \bar{p} \cdot \nabla_\xi (\nabla_\xi p \cdot \nabla \phi). \end{aligned}$$

In particular, evaluating the above at (x_0, ξ), we obtain

$$\begin{aligned} \{\bar{p}, \{p, \phi\}\}(x_0, \xi) &= \nabla_\xi \bar{p}(x_0, \xi) \cdot \nabla^2 \phi(x_0) \cdot \nabla_\xi p(x_0, \xi) \\ &\quad + \sum_{\alpha,\beta=1}^n \partial^2_{x^\alpha \xi_\beta} p(x_0, \xi) \partial_{\xi_\alpha} \bar{p}(x_0, \xi) \partial_{x^\beta} \phi(x_0) \\ &\quad - \sum_{\alpha,\beta=1}^n \partial^2_{\xi_\alpha \xi_\beta} p(x_0, \xi) \partial_{x^\alpha} \bar{p}(x_0, \xi) \partial_{x^\beta} \phi(x_0). \end{aligned}$$

The first part of (4.7) follows from the above, along with the observation that the first term on the right-hand side of the above is always real-valued.

Next, for the second part of (4.7), we directly compute

$$\begin{aligned}
-i\{\overline{p_\phi}, p_\phi\}(x,\xi,\lambda) &= \operatorname{Im} \nabla_\xi \left[\overline{p(x,\xi - i\lambda\, d\phi(x))}\right] \cdot \nabla_x [p(x,\xi - i\lambda\, d\phi(x))] \\
&= \operatorname{Im}[\nabla_\xi \bar{p}(x,\xi - i\lambda\, d\phi(x)) \cdot \nabla_x p(x,\xi - i\lambda\, d\phi(x))] \\
&\qquad - \operatorname{Im}[i\lambda\, \nabla_\xi \bar{p}(x,\xi - i\lambda\, d\phi(x)) \cdot \nabla^2\phi(x) \cdot \nabla_\xi p(x,\xi - i\lambda\, d\phi(x))].
\end{aligned}$$

The desired identity follows, since the following is always real-valued:

$$\nabla_\xi \bar{p}(x,\xi - i\lambda\, d\phi(x)) \cdot \nabla^2\phi(x) \cdot \nabla_\xi p(x,\xi - i\lambda\, d\phi(x))). \qquad \square$$

4.2. **Some Expansions.** We now present some estimates that will be useful for our unique continuation result. We begin with some properties of the weights ϕ_μ from Proposition 2.15.

Definition 4.8. *Fix $\mu_* \geq 1$ large enough so that $U_{\mu_*} \subset\subset \Omega$. Given any $\mu \geq \mu_*$, we set*

$$U_\mu := \{x \in \Omega \mid |x - x_0| < \mu^{-2}\}. \tag{4.8}$$

Proposition 4.9. *Let $\mu \geq \mu_*$, and let ϕ_μ be defined as in* (2.10)*. Then, for any $x \in U_\mu$,*

$$\begin{aligned}
\nabla\phi_\mu(x) &= \nabla\rho(x_0) + \nabla^2\rho(x_0)\cdot(x - x_0) - 2\mu[\nabla\rho(x_0)\cdot(x - x_0)]\nabla\rho(x_0) \\
&\qquad + \mu^{-1}(x - x_0), \\
\nabla^2\phi_\mu(x) &= \nabla^2\rho(x_0) - 2\mu[\nabla\rho(x_0)\otimes\nabla\rho(x_0)] + \mu^{-1} I_n,
\end{aligned} \tag{4.9}$$

where I_n denotes the $n \times n$ identity matrix. Furthermore:

- *There exists a constant $C > 0$ such that for any $x \in U_\mu$,*

$$|\nabla\phi_\mu(x) - \nabla\rho(x_0)| \leq C\mu^{-1}. \tag{4.10}$$

- *There exist constants $C, C' > 0$ such that given any $(x,\xi,\lambda) \in U \times \mathbb{R}^n \times [1,\infty)$, the following inequality holds, with $\zeta_\mu := \xi - i\lambda\, d\phi_\mu(x)$:*

$$\begin{aligned}
&\mu(\lambda^2 + |\xi|^2)^{-\frac{1}{2}} |p_{\phi_\mu}(x,\xi,\lambda)|^2 + \{\operatorname{Re} p_{\phi_\mu}, \operatorname{Im} p_{\phi_\mu}\}(x,\xi,\lambda) \\
&\quad \geq C'\mu|\zeta_\mu|^{-1}|p(x,\zeta_\mu)|^2 + \mu\lambda\, |\nabla_\xi p(x,\zeta_\mu)\cdot\nabla\rho(x_0)|^2 + \tfrac{1}{2}\operatorname{Im}(\nabla_\xi \bar{p}\cdot\nabla_x p)(x,\zeta_\mu) \\
&\qquad - \tfrac{1}{2}\lambda\, \nabla_\xi \bar{p}(x,\zeta_\mu)\cdot\nabla^2\rho(x_0)\cdot\nabla_\xi p(x,\zeta_\mu) - C\mu^{-1}\lambda\, |\zeta_\mu|^{2m-2}.
\end{aligned} \tag{4.11}$$

Proof. First, both identities in (4.9) follow from directly differentiating the defining formula (2.10). Furthermore, (4.10) then follows from (4.8) and the first part of (4.9).

Next, for (4.11), we begin by applying (3.16), (4.5), and the last part of (4.7):

$$\begin{aligned}
&\mu(\lambda^2 + |\xi|^2)^{-\frac{1}{2}} |p_{\phi_\mu}(x,\xi,\lambda)|^2 + \{\operatorname{Re} p_{\phi_\mu}, \operatorname{Im} p_{\phi_\mu}\}(x,\xi,\lambda) \\
&\quad \geq C'\mu|\zeta_\mu|^{-1}|p(x,\zeta_\mu)|^2 + \tfrac{1}{2}\operatorname{Im}(\nabla_\xi \bar{p}\cdot\nabla_x p)(x,\zeta_\mu) - \tfrac{1}{2}\lambda\, \nabla_\xi \bar{p}(x,\zeta_\mu)\cdot\nabla^2\phi_\mu(x)\cdot\nabla_\xi p(x,\zeta_\mu).
\end{aligned}$$

(In the above, we also observed that $|\zeta_\mu|^2 \simeq \lambda^2 + |\xi|^2$ on U_μ.) The result now follows from applying the last part of (4.9) to the above, and noting that

$$|\nabla_\xi \bar{p}(x,\zeta_\mu)|^2 \lesssim |\zeta_\mu|^{2m-2},$$

since the left-hand side of the above is homogeneous of order $2m - 2$ in ζ_μ. $\square$

We will also need the following estimate in our unique continuation result:

Proposition 4.10. *Suppose p is real-valued. Moreover, let $\mu \geq \mu_*$, and suppose ϕ_μ is defined as in* (2.10). *Then, there exists some constant $C > 0$ such that the following inequality holds for all $(x, \xi, \lambda) \in U_\mu \times \mathbb{R}^n \times [1, \infty)$, and with $\zeta_\mu := \xi - i\lambda \, d\phi_\mu(x)$:*

$$\begin{aligned} \operatorname{Im}(\nabla_\xi \bar{p} \cdot \nabla_x p)(x, \zeta_\mu) &- \lambda \, \nabla_\xi \bar{p}(x, \xi) \cdot \nabla^2 \rho(x_0) \cdot \nabla_\xi p(x, \xi) \\ &\geq -\lambda \operatorname{Re}\{\bar{p}, \{p, \rho\}\}(x_0, \xi) - C\mu^{-1}\lambda|\zeta_\mu|^{2m-2} - C\lambda^2|\zeta_\mu|^{2m-3}. \end{aligned} \tag{4.12}$$

Proof. Applying Taylor's theorem about $\lambda = 0$, we have, for $(x, \xi, \lambda) \in U_\mu \times \mathbb{R}^n \times (0, \infty)$,

$$\begin{aligned} \operatorname{Im}(\nabla_\xi \bar{p} \cdot \nabla_x p)(x, \zeta_\mu) &= \operatorname{Im}(\nabla_\xi \bar{p} \cdot \nabla_x p)(x, \xi) + \lambda \, \partial_\tau [\operatorname{Im}(\nabla_\xi \bar{p} \cdot \nabla_x p)(x, \zeta_\mu)]|_{\tau=0} \\ &\qquad + \lambda^2 \, \sigma(x, \xi, \lambda) \\ &= -\lambda \operatorname{Re} \nabla \phi_\mu(x) \cdot \nabla_\xi (\nabla_\xi \bar{p} \cdot \nabla_x p)(x, \xi) + \lambda^2 \, \sigma(x, \xi, \lambda), \end{aligned} \tag{4.13}$$

where we noted $\operatorname{Im}(\nabla_\xi \bar{p} \cdot \nabla_x p) \equiv 0$ due to p being real-valued, and where σ is a homogeneous polynomial in (ξ, λ) of order $2m - 3$ and hence satisfies (as $|\zeta_\mu|^2 \simeq \lambda^2 + |\xi|^2$ and $U_\mu \subset\subset \Omega$)

$$|\sigma(x, \xi, \lambda)| \lesssim |\zeta_\mu|^{2m-3}, \qquad (x, \xi, \lambda) \in U_\mu \times \mathbb{R}^n \times [1, \infty). \tag{4.14}$$

Further expanding the right-hand side of (4.13), we obtain

$$\begin{aligned} \operatorname{Im}(\nabla_\xi \bar{p} \cdot \nabla_x p)(x, \zeta_\mu) &= \lambda \operatorname{Re} \sum_{\alpha, \beta = 1}^n \mathcal{R}_{\alpha\beta}(x_0, \xi) \partial_{x^\beta} \rho(x_0) + \mathcal{E}(x, \xi, \lambda), \\ \mathcal{R}_{\alpha\beta}(x, \xi) &:= (\partial^2_{\xi_\alpha \xi_\beta} \bar{p} \partial_{x^\alpha} p - \partial^2_{x^\alpha \xi_\beta} p \partial_{\xi_\alpha} \bar{p})(x, \xi), \\ \mathcal{E}(x, \xi, \lambda) &:= \lambda \operatorname{Re} \sum_{\alpha, \beta = 1}^n \mathcal{R}_{\alpha\beta}(x, \xi) [\partial_{x^\beta} \phi_\mu(x) - \partial_{x^\beta} \rho(x_0)] + \lambda^2 \, \sigma(x, \xi, \lambda) \\ &\qquad + \lambda \operatorname{Re} \sum_{\alpha, \beta = 1}^n [\mathcal{R}_{\alpha\beta}(x, \xi) - \mathcal{R}_{\alpha\beta}(x_0, \xi)] \partial_{x^\beta} \rho(x_0). \end{aligned} \tag{4.15}$$

Since p is a homogeneous polynomial in ξ of order m, then (4.8), (4.10), and (4.14) yield

$$|\mathcal{E}(x, \xi, \lambda)| \lesssim \mu^{-1}\lambda|\zeta_\mu|^{2m-2} + \lambda^2|\zeta_\mu|^{2m-3}. \tag{4.16}$$

Combining the first part of (4.7), (4.15), and (4.16), we see there exists $C' > 0$ with

$$\begin{aligned} \operatorname{Im}(\nabla_\xi \bar{p} \cdot \nabla_x p)(x, \zeta_\mu) &- \lambda \, \nabla_\xi \bar{p}(x, \xi) \cdot \nabla^2 \rho(x_0) \cdot \nabla_\xi p(x, \xi) \\ &\geq \lambda \, [\nabla_\xi \bar{p}(x_0, \xi) \cdot \nabla^2 \rho(x_0) \cdot \nabla_\xi p(x_0, \xi) - \nabla_\xi \bar{p}(x, \xi) \cdot \nabla^2 \rho(x_0) \cdot \nabla_\xi p(x, \xi)] \\ &\qquad - \lambda \operatorname{Re}\{\bar{p}, \{p, \rho\}\}(x_0, \xi) - C'\mu^{-1}\lambda|\zeta_\mu|^{2m-2} - C'\lambda^2|\zeta_\mu|^{2m-3}. \end{aligned}$$

The desired (4.12) follows immediately from the above after recalling (4.8). ☐

4.3. **The Main Result.** We are now prepared to state and prove our main unique continuation result. In practice, the remaining task is to prove the symbol inequality (3.19).

Theorem 4.11. *Suppose p is real-valued, and assume (Σ, ρ) is strongly pseudoconvex at x_0 with respect to $\mathcal{P}$. Then, $(\mathcal{P}, \Sigma, \rho)$ has the local unique continuation property at x_0.*

Proof. From Theorem 2.22, it suffices to show that $(\mathcal{P}, \Sigma, \rho)$ admits a Carleman estimate at x_0. For sufficiently large $\mu \geq \mu_*$, we define ϕ_μ and U_μ as in (2.10) and (4.8), respectively. Note that Proposition 2.15 implies ϕ_μ is a Carleman weight for ρ at x_0. Thus, by Proposition 3.16, it suffices to show that there exists sufficiently large $\mu_0 \geq \lambda_0 > 0$ such that

$$\mu(\lambda^2 + |\xi|^2)^{-\frac{1}{2}} |p_{\phi_\mu}(x, \xi, \lambda)|^2 + \{\operatorname{Re} p_{\phi_\mu}, \operatorname{Im} p_{\phi_\mu}\}(x, \xi, \lambda) \geq \mu^{-1} \lambda (\lambda^2 + |\xi|^2)^{m-1} \tag{4.17}$$

holds for all $\mu \geq \mu_0$, $x \in U_\mu$, $\xi \in \mathbb{R}^n$, and $\lambda \geq \lambda_0$.

Suppose, for a contradiction, that the inequality (4.17) fails to hold. Then, there are

$$x_k \in U_k, \qquad (\xi_k, \lambda_k) \in \mathbb{R}^n \times [k^2, \infty), \qquad \mathbb{N} \ni k \geq \mu_*, \tag{4.18}$$

such that the following holds for all $k \geq \mu_*$:

$$k(\lambda_k^2 + |\xi_k|^2)^{-\frac{1}{2}} |p_{\phi_k}(x_k, \xi_k, \lambda_k)|^2 + \{\operatorname{Re} p_{\phi_k}, \operatorname{Im} p_{\phi_k}\}(x_k, \xi_k, \lambda_k) < \tfrac{1}{k} \lambda_k (|\xi_k|^2 + \lambda_k^2)^{m-1}. \tag{4.19}$$

For convenience, we define, for each $k \geq \mu_*$, the shorthands

$$\zeta_k := \xi_k - i\lambda_k \, d\phi_k(x_k), \qquad |\zeta_k|^2 \simeq \lambda_k^2 + |\xi_k|^2. \tag{4.20}$$

as well as the normalizations

$$Z_k := |\zeta_k|^{-1} \zeta_k, \qquad \Xi_k := |\zeta_k|^{-1} \xi_k, \qquad \Lambda_k := |\zeta_k|^{-1} \lambda_k. \tag{4.21}$$

Now, since $x_k \in U_k$, then (4.8) and (4.18) imply $|x_k - x_0| < k^{-2}$, and by (4.9),

$$\lim_{k \to \infty} x_k = x_0, \qquad \lim_{k \to \infty} \nabla \phi_k(x_k) = \nabla \rho(x_0). \tag{4.22}$$

Furthermore, passing to a subsequence, we obtain additional limits

$$\lim_{k \to \infty} (\Xi_k, \Lambda_k) = (\Xi_0, \Lambda_0) \neq (0, 0), \qquad \lim_k Z_k = Z_0, \quad |Z_0| = 1. \tag{4.23}$$

Using (4.11) (with $\mu := k$), along with (4.20), we can expand (4.19) as

$$\begin{aligned} C_* k^{-1} \lambda_k |\zeta_k|^{2m-2} \geq{}& C_\dagger k \, |\zeta_k|^{-1} |p(x_k, \zeta_k)|^2 + k \lambda_k \, |\nabla_\xi p(x_k, \zeta_k) \cdot \nabla \rho(x_0)|^2 \\ &+ \tfrac{1}{2} \operatorname{Im}(\nabla_\xi \bar{p} \cdot \nabla_x p)(x_k, \zeta_k) \\ &- \tfrac{1}{2} \lambda_k \, \nabla_\xi \bar{p}(x_k, \zeta_k) \cdot \nabla^2 \rho(x_0) \cdot \nabla_\xi p(x_k, \zeta_k), \end{aligned} \tag{4.24}$$

for some $C_*, C_\dagger > 0$. Dividing (4.24) by $k |\zeta_k|^{2m-1}$ then yields

$$\begin{aligned} C_* \Lambda_k k^{-2} \geq{}& C_\dagger |p(x_k, Z_k)|^2 + \tfrac{1}{2k} \operatorname{Im}(\nabla_\xi \bar{p} \cdot \nabla_x p)(x_k, Z_k) + \Lambda_k \, |\nabla_\xi p(x_k, Z_k) \cdot \nabla \rho(x_0)|^2 \\ &- \tfrac{1}{2k} \Lambda_k \, \nabla_\xi \bar{p}(x_k, Z_k) \cdot \nabla^2 \rho(x_0) \cdot \nabla_\xi p(x_k, Z_k). \end{aligned} \tag{4.25}$$

From here, the proof splits into two cases, depending on the value of Λ_0.

First, suppose $\Lambda_0 > 0$. Letting $k \nearrow \infty$ in (4.25), then (4.22) and (4.23) yield

$$|p(x_0, Z_0)|^2 = 0, \qquad p_\rho(x_0, \Xi_0, \Lambda_0) = 0. \tag{4.26}$$

Dividing (4.24) by $k \lambda_k |\zeta_k|^{2m-2}$ and discarding the non-negative term $|p(x_k, \zeta_k)|^2$, we have

$$\begin{aligned} C_* k^{-2} \geq{}& \tfrac{1}{2} k^{-1} \Lambda_k^{-1} \operatorname{Im}(\nabla_\xi \bar{p} \cdot \nabla_x p)(x_k, Z_k) + |\nabla_\xi p(x_k, Z_k) \cdot \nabla \rho(x_0)|^2 \\ &- \tfrac{1}{2} k^{-1} \, \nabla_\xi \bar{p}(x_k, Z_k) \cdot \nabla^2 \rho(x_0) \cdot \nabla_\xi p(x_k, Z_k). \end{aligned}$$

Letting $k \nearrow \infty$, noting that $\Lambda_k^{-1} \to \Lambda_0^{-1} > 0$, and using (4.22) and (4.23), we see that

$$|\nabla_\xi p(x_0, Z_0) \cdot \nabla \rho(x_0)|^2 = 0, \qquad \{p_\rho, \rho\}(x_0, \Xi_0, \Lambda_0) = 0. \tag{4.27}$$

Now, dropping both non-negative terms in (4.24) and dividing the result by $|\zeta_k|^{2m-1}$ yields

$$C_* k^{-1} \Lambda_k \geq \tfrac{1}{2} \operatorname{Im}(\nabla_\xi \bar{p} \cdot \nabla_x p)(x_k, Z_k) - \tfrac{1}{2} \Lambda_k \nabla_\xi \bar{p}(x_k, Z_k) \cdot \nabla^2 \rho(x_0) \cdot \nabla_\xi p(x_k, Z_k).$$

Taking the limit $k \nearrow \infty$ in the above and recalling (4.22)–(4.23) results in the inequality

$$\operatorname{Im}(\nabla_\xi \bar{p} \cdot \nabla_x p)(x_0, Z_0) - \Lambda_0 \nabla^2 \rho(x_0) \cdot (\nabla_\xi \bar{p}(x_0, Z_0), \nabla_\xi p(x_0, Z_0)) \leq 0.$$

Combining the above with the second part of (4.7) yields

$$-i \{\overline{p_\rho}, p_\rho\}(x_0, \Xi_0, \Lambda_0) \leq 0. \tag{4.28}$$

In particular, (4.26)–(4.28) contradict that (Σ, ρ) strongly pseudoconvex at x_0.

Therefore, it remains only to consider the case $\Lambda_0 = 0$, and to derive a contradiction in this setting. Letting $k \nearrow \infty$ in (4.25)—and recalling (4.22) and (4.23)—now yields

$$p(x_0, \Xi_0) = 0. \tag{4.29}$$

Moreover, applying (4.12) (again with $\mu := k$) to (4.24) yields, for some $C' > 0$,

$$\begin{aligned} C_* k^{-1} \lambda_k |\zeta_k|^{2m-2} &\geq C_\dagger k \, |\zeta_k|^{-1} |p(x_k, \zeta_k)|^2 + k \lambda_k \, |\nabla_\xi p(x_k, \zeta_k) \cdot \nabla \rho(x_0)|^2 \\ &\quad - \tfrac{1}{2} \lambda_k \operatorname{Re}\{\bar{p}, \{p, \rho\}\}(x_0, \xi_k) - C' k^{-1} \lambda_k |\zeta_k|^{2m-2} - C' \lambda_k^2 |\zeta_k|^{2m-3}, \end{aligned} \tag{4.30}$$

Dividing (4.30) by $k \lambda_k |\zeta_k|^{2m-2}$, we then obtain

$$C_* k^{-2} \geq |\nabla_\xi p(x_k, Z_k) \cdot \nabla \rho(x_0)|^2 - \tfrac{1}{2} k^{-1} \operatorname{Re}\{\bar{p}, \{p, \rho\}\}(x_k, \Xi_k) - C' k^{-2} - C' \Lambda_k k^{-1}.$$

Letting $k \nearrow \infty$ in the above, applying (4.22) and (4.23), and recalling that $\Lambda_0 = 0$ yields

$$|\nabla_\xi p(x_0, \Xi_0) \cdot \nabla \rho(x_0)|^2 = 0, \qquad \{p, \rho\}(x_0, \Xi_0) = 0. \tag{4.31}$$

We now discard the first two terms on the right-hand side of (4.30) and divide by $\lambda_k |\zeta_k|^{2m-2}$:

$$C_* k^{-1} \geq -\tfrac{1}{2} \operatorname{Re}\{\bar{p}, \{p, \rho\}\}(x_k, \Xi_k) - C' k^{-1} - C' \Lambda_k.$$

Letting $k \nearrow \infty$ in the above and recalling that $\Lambda_0 = 0$ results in the inequality

$$-\operatorname{Re}\{\bar{p}, \{p, \rho\}\}(x_0, \Xi_0) \leq 0. \tag{4.32}$$

Since $\Lambda_0 = 0$, then (4.23) implies $\Xi_0 \neq 0$, hence (4.29), (4.31), and (4.32) again contradict the assumption that (Σ, ρ) is strongly pseudoconvex at x_0.

Thus, we have a contradiction for all possible cases, completing the proof of (4.17). □

5. Second-Order Operators

In this section, we apply our theory to the special case of second-order operators with purely real principal symbols. This includes both elliptic equations arising from Riemannian metrics and wave equations arising from Lorentzian metrics.

5.1. Classification of Operators.

We begin by defining our notations:

Remark 5.1. *For convenience, we adopt Einstein summation notation—indices repeated in superscript and subscript are understood to be summed over the values* $1, \dots, n$. *Furthermore, we use the standard shorthand* ∂_α *to represent the derivative* ∂_{x^α}.

Assumption 5.2. *Let* $\mathcal{G}$ *denote the second-order linear partial differential operator*

$$\begin{aligned} \mathcal{G} &:= g^{\alpha\beta}\, \partial_\alpha \partial_\beta + g^\alpha\, \partial_\alpha + g^\circ \\ &= -g^{\alpha\beta}\, D_\alpha D_\beta + i\, g^\alpha\, D_\alpha + g^\circ, \end{aligned} \tag{5.1}$$

where the second-order coefficients satisfy

$$g^{\alpha\beta} \in C^\infty(\Omega; \mathbb{R}), \qquad g^{\alpha\beta} = g^{\beta\alpha}, \quad \alpha, \beta \in \{1, \dots, n\}, \tag{5.2}$$

and where the lower-order coefficients satisfy

$$g^\alpha, g^\circ \in C^\infty(\Omega; \mathbb{C}), \qquad \alpha \in \{1, \dots, n\}.$$

Furthermore, keeping with our conventions, we let g *denote the principal symbol of* $\mathcal{G}$:

$$g(x, \xi) = -g^{\alpha\beta}(x)\, \xi_\alpha \xi_\beta, \qquad (x, \xi) \in \Omega \times \mathbb{R}^n. \tag{5.3}$$

Remark 5.3. *The symmetry assumption in* (5.2) *does not result in any loss of generality; one can always reduce to the symmetric case by replacing each* $g^{\alpha\beta}$ *by* $\frac{1}{2}(g^{\alpha\beta} + g^{\beta\alpha})$.

Remark 5.4. *We can associate the principal coefficients* $g^{\alpha\beta}$ *with a (co-)metric* $\mathbf{g}$ *on* Ω:

$$\mathbf{g} : \Omega \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \qquad \mathbf{g}(x; \xi, \zeta) = g^{\alpha\beta}(x)\, \xi_\alpha \zeta_\beta. \tag{5.4}$$

In particular, observe that:

- *If* $\mathbf{g}$ *is Riemannian, then* $\mathcal{G}$ *is a geometric Laplace-Beltrami operator.*
- *If* $\mathbf{g}$ *is Lorentzian, then* $\mathcal{G}$ *is a geometric wave operator.*

Proposition 5.5. *The following properties hold:*

- $\mathcal{G}$ *is elliptic at* x_0 *if and only if* $g(x_0, \cdot)$ *is everywhere positive or everywhere negative.*
- Σ *is non-characteristic at* x_0 *with respect to* $\mathcal{G}$ *if and only if*

$$(g^{\alpha\beta} \partial_\alpha \rho \partial_\beta \rho)(x_0) \neq 0. \tag{5.5}$$

Proof. This follows immediately from Definition 2.9, Definition 2.10, and (5.3). □

Remark 5.6. *Proposition 5.5 can also be reformulated using geometric language:*

- $\mathcal{G}$ *is elliptic at* x_0 *if and only if* $+\mathbf{g}$ *or* $-\mathbf{g}$ *is Riemannian at* x_0.
- Σ *is non-characteristic at* x_0 *with respect to* $\mathcal{G}$ *if and only if* Σ *is not* $\mathbf{g}$*-null at* x_0.

Lemma 5.7. *Let $\xi \in \mathbb{R}^n$, and assume ξ is of the form*

$$\xi = \xi' + a\, d\rho(x_0), \qquad a \in \mathbb{R}, \quad (g^{\alpha\beta} \partial_\alpha \rho)(x_0)\, \xi'_\beta = 0. \tag{5.6}$$

Then, the polynomial

$$f_{g,x_0,\xi} : \mathbb{C} \to \mathbb{C}, \qquad f_{g,x_0,\xi}(\sigma) := g(x_0, \xi + \sigma\, d\rho(x_0)) \tag{5.7}$$

has simple, real roots if and only if

$$(g^{\alpha\beta} \partial_\alpha \rho \partial_\beta \rho)(x_0) \cdot g^{\mu\nu}(x_0)\, \xi'_\mu \xi'_\nu < 0. \tag{5.8}$$

Proof. Using (5.3), (5.6), and (5.7), we expand, for any $\sigma \in \mathbb{C}$,

$$\begin{aligned} f_{g,x_0,\xi}(\sigma) &= -g^{\alpha\beta}(x_0)(\xi_\alpha + \sigma\, \partial_\alpha \rho(x_0))(\xi_\beta + \sigma\, \partial_\beta \rho(x_0)) \\ &= -(g^{\alpha\beta} \partial_\alpha \rho \partial_\beta \rho)(x_0) \cdot \sigma^2 - 2(g^{\alpha\beta} \partial_\alpha \rho)(x_0) \xi_\beta \cdot \sigma - g^{\alpha\beta}(x_0) \xi_\alpha \xi_\beta \\ &= -(g^{\alpha\beta} \partial_\alpha \rho \partial_\beta \rho)(x_0) \cdot \sigma^2 - 2a(g^{\alpha\beta} \partial_\alpha \rho \partial_\beta \rho)(x_0) \cdot \sigma \\ &\qquad - [g^{\alpha\beta}(x_0) \xi'_\alpha \xi'_\beta + a^2 (g^{\alpha\beta} \partial_\alpha \rho \partial_\beta \rho)(x_0)]. \end{aligned}$$

Then, $f_{g,x_0,\xi}$ has simple, real roots if and only if its discriminant is strictly positive:

$$\begin{aligned} 0 &< a^2 [(g^{\alpha\beta} \partial_\alpha \rho \partial_\beta \rho)(x_0)]^2 - (g^{\alpha\beta} \partial_\alpha \rho \partial_\beta \rho)(x_0) [g^{\alpha\beta}(x_0) \xi'_\alpha \xi'_\beta + a^2 (g^{\alpha\beta} \partial_\alpha \rho \partial_\beta \rho)(x_0)] \\ &= -(g^{\alpha\beta} \partial_\alpha \rho \partial_\beta \rho)(x_0) \cdot g^{\alpha\beta}(x_0) \xi'_\alpha \xi'_\beta. \end{aligned}$$

□

Proposition 5.8. *Σ is strictly hyperbolic at x_0 with respect to $\mathcal{G}$ if and only if*

$$(g^{\alpha\beta} \partial_\alpha \rho \partial_\beta \rho)(x_0) \cdot g^{\mu\nu}(x_0)\, \xi'_\mu \xi'_\nu < 0 \tag{5.9}$$

holds for any $\xi' \in \mathbb{R}^n \setminus \{0\}$ satisfying

$$(g^{\alpha\beta} \partial_\alpha \rho)(x_0)\, \xi'_\beta = 0. \tag{5.10}$$

Proof. First, assume Σ is strictly hyperbolic at x_0 with respect to $\mathcal{G}$. If $\xi' \in \mathbb{R}^n \setminus \{0\}$ satisfies (5.10), then $\xi' \wedge d\rho(x_0) \neq 0$, since otherwise this implies

$$(g^{\alpha\beta} \partial_\alpha \rho \partial_\beta \rho)(x_0) = 0,$$

which by Proposition 5.5 contradicts that Σ is non-characteristic at x_0. By Definition 2.11, the polynomial $f_{g,x_0,\xi'}$ (as defined in (5.7), with $a := 0$) has simple and real roots, and hence the desired (5.9) follows immediately from Lemma 5.7.

Conversely, assume that (5.9) holds for all $\xi' \in \mathbb{R}^n \setminus \{0\}$ satisfying (5.10). First, we claim that (5.5) must hold. This is a consequence of the following:

- If there exists $\xi' \in \mathbb{R}^n \setminus \{0\}$ such that (5.10) holds, then (5.5) follows from (5.9).
- Otherwise, if no such $\xi' \in \mathbb{R}^n \setminus \{0\}$ exists, then (5.5) trivially holds.

In particular, Proposition 5.5 implies Σ is non-characteristic at x_0 with respect to $\mathcal{G}$.

Now, fix $\xi \in \mathbb{R}^n$ with $\xi \wedge d\rho(x_0) \neq 0$. Using (5.5), we obtain that ξ is of the form (5.6) for some $a \in \mathbb{R}$. Moreover, the condition $\xi \wedge d\rho(x_0) \neq 0$ implies $\xi' \neq 0$, and hence (5.9) holds with this ξ'. By Lemma 5.7, we conclude that the polynomial $f_{g,x_0,\xi}$ (with the above ξ) must have simple, real roots. This proves that Σ is strictly hyperbolic at x_0. □

Remark 5.9. *Proposition 5.8 also has an equivalent geometric formulation. When $n > 1$, we have that Σ is strictly hyperbolic at x_0 with respect to $\mathcal{G}$ if and only if $\mathbf{g}$ is Lorentzian at x_0 (with either $(-, +, \dots, +)$ or $(+, -, \dots, -)$ signature) and Σ is $\mathbf{g}$-spacelike at x_0.*

Remark 5.10. *Furthermore, in the trivial case $n = 1$, we obtain from Proposition 5.8 that Σ is strictly hyperbolic at x_0 with respect to $\mathcal{G}$ if and only if $\mathbf{g}$ is non-degenerate at x_0.*

Finally, for $\phi \in C^\infty(\Omega; \mathbb{R})$, we apply Definition 3.13 to $\mathcal{G}$ and g:

$$\begin{aligned} g_\phi(x, \xi, \lambda) &= -g^{\alpha\beta}[\xi_\alpha - i\lambda\, \partial_\alpha \phi(x)][\xi_\beta - i\lambda\, \partial_\beta \phi(x)] \\ &= -g^{\alpha\beta}(x)\, \xi_\alpha \xi_\beta + 2i\lambda\, (g^{\alpha\beta}\partial_\alpha \phi)(x)\, \xi_\beta + \lambda^2\, (g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi)(x), \end{aligned} \tag{5.11}$$

for any $(x, \xi, \lambda) \in \Omega \times \mathbb{R}^n \times \mathbb{R}$. In particular, its real and imaginary parts are

$$\begin{aligned} \operatorname{Re} g_\phi(x, \xi, \lambda) &= -g^{\alpha\beta}(x)\, \xi_\alpha \xi_\beta + \lambda^2\, (g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi)(x), \\ \operatorname{Im} g_\phi(x, \xi, \lambda) &= 2\lambda\, (g^{\alpha\beta}\partial_\alpha \phi)(x)\, \xi_\beta. \end{aligned} \tag{5.12}$$

5.2. **Pseudoconvexity and Unique Continuation.** We now study how the strong pseudoconvexity condition simplifies in the special case of our operator $\mathcal{G}$. For this, we apply the calculations in the previous section to the principal symbol g.

Definition 5.11. *Given $\psi \in C^\infty(\Omega; \mathbb{C})$, we define the g-Hessian $\nabla_g \psi$ of ψ to be the matrix-valued function on Ω whose components are given by*

$$(\nabla_g^2 \psi)^{\alpha\beta} = g^{\alpha\mu} g^{\beta\nu} \partial_\mu \partial_\nu \psi + \tfrac{1}{2}(g^{\alpha\mu}\partial_\mu g^{\beta\nu} + g^{\beta\mu}\partial_\mu g^{\alpha\nu} - g^{\mu\nu}\partial_\mu g^{\alpha\beta})\partial_\nu \psi. \tag{5.13}$$

Remark 5.12. *$\nabla_g^2 \psi$ also has a geometric interpretation—when the corresponding metric $\mathbf{g}$ is non-degenerate, $\nabla_g^2 \psi$ is precisely the (dual of the) Levi-Civita $\mathbf{g}$-covariant Hessian of g.*

Proposition 5.13. *The following hold for any $\phi \in C^\infty(\Omega; \mathbb{R})$, $\xi \in \mathbb{R}^n$, and $\lambda \in (0, \infty)$:*

$$\begin{aligned} \{g, \phi\}(x_0, \xi) &= -2(g^{\alpha\beta}\partial_\alpha \phi)(x_0)\, \xi_\beta \\ -\operatorname{Re}\{\bar{g}, \{g, \phi\}\}(x_0, \xi) &= -4\, \xi \cdot \nabla_g^2 \phi(x_0) \cdot \xi, \\ -i\{\overline{g_\phi}, g_\phi\}(x_0, \xi, \lambda) &= -4\lambda\, \xi \cdot \nabla_g^2\phi(x_0) \cdot \xi - 4\lambda^3\, \nabla\phi(x_0) \cdot \nabla_g^2 \phi(x_0) \cdot \nabla\phi(x_0). \end{aligned} \tag{5.14}$$

Proof. First, note that direct computations using (5.11) yield that

$$\begin{aligned} \partial_{\xi_\alpha} g(x_0, \xi) &= -2g^{\alpha\beta}(x_0)\, \xi_\beta, & \partial_{x^\alpha} g(x_0, \xi) &= -\partial_\alpha g^{\mu\nu}(x_0)\, \xi_\mu \xi_\nu, \\ \partial_{\xi_\alpha}\partial_{x^\beta} g(x_0, \xi) &= -2\partial_{x^\beta} g^{\alpha\mu}(x_0)\, \xi_\mu, & \partial_{\xi_\alpha}\partial_{\xi_\beta} g(x_0, \xi) &= -2g^{\alpha\beta}(x_0). \end{aligned} \tag{5.15}$$

The first part of (5.14) now follows from the above and Definition 3.7.

Next, for the second part of (5.14), we first apply (5.15) to obtain

$$-\nabla_\xi \bar{g}(x_0, \xi) \cdot \nabla^2 \phi(x_0) \cdot \nabla_\xi g(x_0, \xi) = -4g^{\alpha\mu} g^{\beta\nu} \partial_\mu \partial_\nu \phi(x_0)\, \xi_\alpha \xi_\beta. \tag{5.16}$$

Additional direct computations using (5.15) yield

$$\begin{aligned} -\operatorname{Re} \partial^2_{x^\alpha \xi_\beta} g(x_0, \xi) \partial_{\xi_\alpha} \bar{g}(x_0, \xi) \partial_{x^\beta} \phi(x_0) &= -4(g^{\alpha\nu}\partial_\alpha g^{\beta\mu}\, \partial_\beta \phi)(x_0)\, \xi_\mu \xi_\nu, \\ \operatorname{Re} \partial^2_{\xi_\alpha \xi_\beta} g(x_0, \xi) \partial_{x^\alpha} \bar{g}(x_0, \xi) \partial_{x^\beta} \phi(x_0) &= 2\, (g^{\alpha\beta}\partial_\alpha g^{\mu\nu} \partial_\beta \phi)(x_0)\, \xi_\mu \xi_\nu. \end{aligned} \tag{5.17}$$

Combining the first part of (4.7), (5.16), and (5.17) results in second part of (5.14) (after some additional algebraic manipulations and reshuffling of repeated indices).

Finally, for the last part of (5.14), we begin by applying (5.15) to compute

$$\begin{aligned}\operatorname{Im}[\nabla_\xi \bar{g}(x_0, \xi - i\lambda\, d\phi(x_0)) \cdot \nabla_x g(x_0, \xi - i\lambda\, d\phi(x_0))] \\ = 2\operatorname{Im}[(g^{\alpha\beta}\partial_\alpha g^{\mu\nu})(x_0)\,(\xi_\beta + i\lambda\,\partial_\beta\phi(x_0))(\xi_\mu - i\lambda\,\partial_\mu\phi(x_0))(\xi_\nu - i\lambda\,\partial_\nu\phi(x_0))] \\ = 2\lambda\,(g^{\alpha\beta}\partial_\alpha g^{\mu\nu})(x_0)\,[\partial_\beta\phi(x_0)\,\xi_\mu\xi_\nu - 2\partial_\mu\phi(x_0)\,\xi_\beta\xi_\nu] \\ - 2\lambda^3\,(g^{\alpha\beta}\partial_\alpha g^{\mu\nu}\,\partial_\beta\phi\partial_\mu\phi\partial_\nu\phi)(x_0).\end{aligned} \tag{5.18}$$

Similarly, using that ∇^2 is symmetric, we can expand

$$\begin{aligned}-\lambda\,\nabla_\xi\bar{g}(x_0, \xi - i\lambda\, d\phi(x_0)) \cdot \nabla^2\phi(x_0) \cdot \nabla_\xi g(x_0, \xi - i\lambda\, d\phi(x_0)) \\ = -4\lambda\,(g^{\alpha\beta}g^{\mu\nu}\,\partial_{\alpha\mu}\phi)(x_0)\,(\xi_\beta + i\lambda\,\partial_\beta\phi(x_0), \xi_\nu - i\lambda\,\partial_\nu\phi(x_0)) \\ = -4\lambda\,(g^{\alpha\beta}g^{\mu\nu}\,\partial_{\alpha\mu}\phi)(x_0)\,\xi_\beta\xi_\nu - 4\lambda^3\,(g^{\alpha\beta}g^{\mu\nu}\,\partial_{\alpha\mu}\phi\partial_\beta\phi\partial_\nu\phi)(x_0).\end{aligned} \tag{5.19}$$

Combining the second part of (4.7), (5.18), and (5.19) yields the last part of (5.14). □

The upshot of our computations is summarized in the following proposition, which shows that for $\mathcal{G}$, strong pseudoconvexity is equivalent to pseudoconvexity. Therefore, the crucial criterion for unique continuation can be simplified considerably for wave equations.

Proposition 5.14. *The following conditions are equivalent:*

- (Σ, ρ) *is strongly pseudoconvex at* x_0 *with respect to* $\mathcal{G}$*.*
- (Σ, ρ) *is pseudoconvex at* x_0 *with respect to* $\mathcal{G}$*.*
- *Given any* $\xi \in \mathbb{R}^n \setminus \{0\}$*, if*

$$g^{\alpha\beta}(x_0)\,\xi_\alpha\xi_\beta = 0, \qquad (g^{\alpha\beta}\partial_\alpha\rho)(x_0)\,\xi_\beta = 0, \tag{5.20}$$

then the following inequality also holds:

$$-\xi \cdot \nabla_g^2\rho(x_0) \cdot \xi > 0. \tag{5.21}$$

Proof. That the second and third statements are equivalent follows from Definition 4.1 and Proposition 5.13. By Definition 4.2, it suffices to show the second statement implies the first. Suppose now that (Σ, ρ) is pseudoconvex at x_0 with respect to $\mathcal{G}$. Then, it suffices to show that if (4.3) holds (with $p := g$) for some $\xi \in \mathbb{R}^n$ and $\lambda > 0$, then so must (4.4).

First, let us suppose $\xi \neq 0$. Note that (4.3) can be equivalently rewritten as

$$g(x_0, \xi - i\lambda\, d\rho(x_0)) = 0, \qquad \partial_\tau[g(x_0, \xi - i\tau\, d\rho(x_0))]|_{\tau=\lambda} = 0. \tag{5.22}$$

This implies $i\lambda$ is a double root of the polynomial $f_{g,x_0,\xi}$ defined in (5.7) (with our present ξ). However, since g is real-valued, then $-i\lambda$ must also be a double root of this polynomial, which yields a contradiction. As a result, (4.3) cannot hold whenever $\xi \neq 0$.

Next, suppose instead that $\xi = 0$. Then, (5.11) and the first part of (5.14) imply that the condition (4.3), in the case $\xi = 0$, is equivalent to

$$g(x_0, d\rho(x_0)) = 0, \qquad \{g, \rho\}(x_0, d\rho(x_0)) = 0. \tag{5.23}$$

Since (Σ, ρ) is pseudoconvex at x_0, then Definition 4.1 and the second part of (5.14) yield

$$-d\rho(x_0) \cdot \nabla_g^2 \rho(x_0) \cdot d\rho(x_0) > 0.$$

Moreover, by the last part of (5.14), the above is precisely the condition (4.4):

$$-i\{\overline{g_\rho}, g_\rho\}(x_0, 0, \lambda) > 0, \qquad \{\operatorname{Re} g_\rho, \operatorname{Im} g_\rho\}(x_0, 0, \lambda) > 0.$$

Finally, combining both cases above completes the proof of strong pseudoconvexity. □

Corollary 5.15. *If $\mathcal{G}$ is elliptic at x_0, then:*

- *(Σ, ρ) is pseudoconvex at x_0 with respect to $\mathcal{G}$.*
- *(Σ, ρ) is strongly pseudoconvex at x_0 with respect to $\mathcal{G}$.*

Proof. By Proposition 5.14, it suffices to show that (Σ, ρ) is pseudoconvex at x_0. However, g being elliptic immediately implies that (4.1) (with g in the place of p) cannot hold. □

Remark 5.16. *Pseudoconvexity also has a geometric interpretation. The conditions* (5.20) *can be reformulated as ξ being $\mathbf{g}$-null and ξ being ($\mathbf{g}$-)cotangent to Σ, respectively. Therefore, (Σ, ρ) being (strongly) pseudoconvex at x_0 can be geometrically characterised as* (5.21) *holding for every $\mathbf{g}$-null covector $\xi \in \mathbb{R}^n \setminus \{0\}$ that is ($\mathbf{g}$-)cotangent to Σ.*

Corollary 5.17. *Suppose that (Σ, ρ) is pseudoconvex at x_0 with respect to $\mathcal{G}$. Then, $(\mathcal{G}, \Sigma, \rho)$ has the local unique continuation property at x_0.*

Proof. Since g is real-valued, and since Proposition 5.14 yields that (Σ, ρ) is strongly pseudoconvex at x_0, the conclusion then follows immediately from Theorem 4.11. □

Appendix A. Microlocal Analysis

In this appendix, we provide self-contained proofs of the key microlocal analysis results from the main sections—in particular, Proposition 3.6, Proposition 3.10, and Theorem 3.11. To keep the discussion as concise as possible, we try to cover only the material needed for the above results, and we leave more systematic expositions to other sources.

Remark A.1. *Some of the material below is an adaptation of parts of Peter Hintz's lecture notes* [1] *to a wider class of λ-parametrized symbols. The proof of the sharp Gårding inequality given here is a special case adapted from the paper* [4] *of Nagase.*

A.1. **Generalized Symbols.** In order to prove the above-mentioned results, we first need to further expand our class of λ-parametrized symbols. The first extension is to generalize our symbols to allow for more flexible derivative estimates:

Definition A.2. *For convenience, we adapt the following shorthand from* (3.6)*:*

$$\gamma : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}, \qquad \gamma(\xi,\lambda) := (\lambda^2 + |\xi|^2)^{\frac{1}{2}}. \tag{A.1}$$

Definition A.3. *Given $s \in \mathbb{R}$ and $\delta \in [0,1)$, we let $\Lambda^s_\delta(\mathbb{R}^n)$ denote the space of all symbols*

$$b \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times (0,\infty); \mathbb{C})$$

satisfying that for any multi-indices I, J, there exists $C > 0$ such that

$$|\nabla_{x,I} \nabla_{\xi,J} b(x,\xi,\lambda)| \leq C \gamma(\xi,\lambda)^{s-|J|+\delta|I|} \tag{A.2}$$

for any $(x,\xi,\lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times [1,\infty)$.

Remark A.4. *Note that $\Lambda^s_0(\mathbb{R}^n)$ coincides with $\Lambda^s(\mathbb{R}^n)$ from Definition 3.1.*

There are multiple ways to associate our symbols with pseudodifferential operators:

Definition A.5. *Given $s \in \mathbb{R}$, $\delta \in [0,1)$, $b \in \Lambda^s_\delta(\mathbb{R}^n)$, and $\lambda \in [1,\infty)$:*

- *The* left quantization $\operatorname{op}^L_\lambda(b)$ *of b is defined, for any $\psi \in C^\infty_0(\mathbb{R}^n;\mathbb{C})$, by*

$$\operatorname{op}^L_\lambda(b)\psi(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} b(x,\xi,\lambda)\psi(y)\, dy d\xi, \qquad x \in \mathbb{R}^n. \tag{A.3}$$

- *The* right quantization $\operatorname{op}^R_\lambda(b)$ *of b is defined, for any $\psi \in C^\infty_0(\mathbb{R}^n;\mathbb{C})$, by*

$$\operatorname{op}^R_\lambda(b)\psi(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} b(y,\xi,\lambda)\psi(y)\, dy d\xi, \qquad x \in \mathbb{R}^n. \tag{A.4}$$

Remark A.6. *Observe that if $b \in \Lambda^s_0(\mathbb{R}^n)$, then the left quantization $\operatorname{op}^L_\lambda(b)$ coincides with $\operatorname{op}_\lambda(b)$ from Definition 3.3 due to the Fourier inversion theorem.*

It will be useful to unify left and right quantizations in our upcoming analysis. For this, we also consider a larger class of symbols depending on two spatial variables:

Definition A.7. *Given $s \in \mathbb{R}$ and $\delta \in [0,1)$, we let $\bar\Lambda^s_\delta(\mathbb{R}^n)$ denote the set of all*

$$b \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times (0,\infty); \mathbb{C})$$

such that for any multi-indices I, J, K, there exists $C > 0$ such that

$$|\nabla_{x,I}\nabla_{y,J}\nabla_{\xi,K} b(x,y,\xi,\lambda)| \leq C\gamma(\xi,\lambda)^{s-|K|+\delta(|I|+|J|)} \tag{A.5}$$

for any $(x, y, \xi, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times [1, \infty)$.

Remark A.8. *As a convention, we will generally refer to the first and second arguments of any symbol $b \in \bar{\Lambda}^s_\delta(\mathbb{R}^n)$ as the "x" and "y" components, respectively.*

Definition A.9. *Given $s \in \mathbb{R}$, $\delta \in [0, 1)$, $b \in \bar{\Lambda}^s_\delta(\mathbb{R}^n)$, and $\lambda \in [1, \infty)$, we define the operator* $\mathrm{Op}_\lambda(b)$ *such that it maps any $\psi \in C^\infty_0(\mathbb{R}^n; \mathbb{C})$ to the function given by*

$$\mathrm{Op}_\lambda(b)\psi(x) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} b(x,y,\xi,\lambda)\psi(y)\,dyd\xi, \qquad x \in \mathbb{R}^n. \tag{A.6}$$

Remark A.10. *Observe that if b in Definition A.9 is independent of y, then* $\mathrm{Op}_\lambda(b)$ *reduces to the left quantization* $\mathrm{op}^L_\lambda(b)$. *On the other hand, if b in Definition A.9 is independent of x, then* $\mathrm{Op}_\lambda(b)$ *reduces to the right quantization* $\mathrm{op}^R_\lambda(b)$.

Observe that all the above quantizations indeed yield well-defined functions:

Proposition A.11. *Fix $s \in \mathbb{R}$, $\delta \in [0, 1)$, and $b \in \bar{\Lambda}^s_\delta(\mathbb{R}^n)$. Then,* $\mathrm{Op}_\lambda(b)\psi$ *is a well-defined complex-valued function on $\mathbb{R}^n$ for any $\psi \in C^\infty_0(\mathbb{R}^n; \mathbb{C})$ and $\lambda \in [1, \infty)$.*

Proof. Since ψ is compactly supported, then recalling the identity

$$(1 + |\xi|^2)^{-N}(1 - \Delta_y)^N e^{i(x-y)\cdot\xi} = e^{i(x-y)\cdot\xi}$$

and integrating the right-hand side of (A.6) by parts in y, we obtain, for any $x \in \mathbb{R}^n$,

$$\begin{aligned}\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} b(x,y,\xi,\lambda)\psi(y)\,dyd\xi\right| &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \tfrac{|(1-\Delta_y)^N[b(x,y,\xi,\lambda)\psi(y)]|}{(1+|\xi|^2)^N}\,dyd\xi \\ &\leq C'\int_{\mathbb{R}^n} \tfrac{\gamma(\xi,\lambda)^{s+2\delta N}}{\gamma(\xi,1)^{2N}}\,d\xi \sum_{|I|\leq 2N}\int_{\mathbb{R}^n} |\nabla^I\psi(y)|\,dy,\end{aligned}$$

for some constant C', where in the last step, we recalled the bounds (A.5) for b.

The integral in y on the right-hand side is finite since ψ has compact support. Since $\delta < 1$, the integral in ξ is also finite as long as N is taken sufficiently large so that $s-2(1-\delta)N < -n$. Thus, by (A.6), it follows that $\mathrm{op}_\lambda(b)\psi(x)$ is well-defined for each $x \in \mathbb{R}^n$. $\square$

Finally, we briefly touch upon the subclass of "infinitely regularizing" operators:

Definition A.12. *Given $\delta \in [0, 1)$, a family of operators $(A_\lambda)_{\lambda\in[1,\infty)}$ mapping $C^\infty_0(\mathbb{R}^n; \mathbb{C})$ to complex-valued functions on $\mathbb{R}^n$ is* δ-residual *iff for any $s \in \mathbb{R}$, there exists $b \in \bar{\Lambda}^s_\delta(\mathbb{R}^n)$ with*

$$A_\lambda = \mathrm{Op}_\lambda(b), \qquad \lambda \in [1, \infty). \tag{A.7}$$

Proposition A.13. *Let $\delta \in [0, 1)$, and let $(A_\lambda)_{\lambda\in[1,\infty)}$ be a δ-residual family of operators. Then, for each $\lambda \in [1, \infty)$, there exists $K_\lambda \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{C})$ such that*

$$A_\lambda\psi(x) = \int_{\mathbb{R}^n} K_\lambda(x,y)\psi(y)\,dy, \qquad \psi \in C^\infty_0(\mathbb{R}^n; \mathbb{C}), \quad (x,\lambda) \in \mathbb{R}^n \times [1,\infty). \tag{A.8}$$

In addition, given any integer $M \geq 0$ and multi-indices I, J, there exists $C > 0$ such that

$$|\nabla_{x,I} \nabla_{y,J} K_\lambda(x,y)| \leq C(\lambda^2 + |x-y|^2)^{-M}, \qquad (x,y,\lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times [1,\infty). \tag{A.9}$$

Proof. Given any $s \in (-\infty, -n)$ and $b \in \bar{\Lambda}^s_\delta(\mathbb{R}^n)$ for which (A.8) holds, the function

$$K_\lambda(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} b(x,y,\xi,\lambda)\, d\xi, \qquad x,y \in \mathbb{R}^n, \tag{A.10}$$

is both well-defined and, by (A.6), satisfies (A.8). Moreover, the identity (A.8) implies that the functions K_λ, $\lambda \in [1,\infty)$, in (A.10) are independent of the choice of s and b.

Letting $s < -n - |I| - |J| - 2M$ and b be as before, then for $\lambda \in [1,\infty)$, we have

$$\begin{aligned} \nabla_{x,I} \nabla_{y,J} K_\lambda(x,y) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} b_{I,J}(x,y,\xi,\lambda)\, d\xi, \\ b_{I,J}(x,y,\xi,\lambda) &= \sum_{I'+I''=I} \sum_{J'+J''=J} (i\xi)_{I'} (-i\xi)_{J'} \nabla_{x,I''} \nabla_{y,J''} b(x,y,\xi,\lambda). \end{aligned} \tag{A.11}$$

(Note the integral in (A.11) is well-defined by (A.5) and our condition on s, which implies $b_{I,J} \in \bar{\Lambda}^{s'}_\delta(\mathbb{R}^n)$ for some $s' < -n - 2M$.) Now, we again recall (A.5) and integrate (A.11) by parts to obtain, for any $x, y \in \mathbb{R}^n$, any $\lambda \in [1,\infty)$, as well as for some constant $C > 0$,

$$\begin{aligned} |\nabla_{x,I} \nabla_{y,J} K_\lambda(x,y)| &= \frac{1}{(2\pi)^n} \left| \int_{\mathbb{R}^n} \tfrac{(\lambda^2 - \Delta_\xi)^M e^{i(x-y)\cdot\xi}}{(\lambda^2 + |x-y|^2)^M} b_{I,J}(x,y,\xi,\lambda)\, d\xi \right| \\ &\leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (\lambda^2 + |x-y|^2)^{-M} |(\lambda^2 - \Delta_\xi)^M b_{I,J}(x,y,\xi,\lambda)|\, d\xi \\ &\leq C(\lambda^2 + |x-y|^2)^{-M}. \end{aligned}$$

□

Remark A.14. *The function K_λ from* (A.8) *is called the* Schwartz kernel *associated to the operator A_λ. Most crucially, note the constants C in* (A.9) *are independent of λ.*

A.2. **Symbol Reductions.** Although Definition A.7 expanded our symbol class, below we show that his has not actually enlarged our class of pseudodifferential operators—that is, our extended symbols produce the same operators as before.

Lemma A.15. *Let $\delta \in [0,1)$, and let $(A_\lambda)_{\lambda \in [1,\infty)}$ be a δ-residual family of operators. Then there exist $r_L, r_R \in \bigcap_{s \in \mathbb{R}} \Lambda^s_\delta(\mathbb{R}^n)$ such that the following hold:*

$$A_\lambda = \mathrm{op}^L_\lambda(r_L) = \mathrm{op}^R_\lambda(r_R), \qquad \lambda \in [1,\infty). \tag{A.12}$$

Proof. Throughout the proof, we let C' denote various positive constants whose values can change between lines but must remain independent of the parameter λ. Also, let $(K_\lambda)_{\lambda \in [1,\infty)}$ denote the Schwartz kernels associated with the family $(A_\lambda)_{\lambda \in [1,\infty)}$, as in Proposition A.13. We now define $r_L, r_R : \mathbb{R}^n \times \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ to satisfy, for $\lambda \in [1,\infty)$,

$$\begin{aligned} r_L(x,\xi,\lambda) &:= \int_{\mathbb{R}^n} e^{-iz\cdot\xi} K_\lambda(x, x-z)\, dz, \\ r_R(y,\xi,\lambda) &:= \int_{\mathbb{R}^n} e^{-iz\cdot\xi} K_\lambda(y+z, y)\, dz. \end{aligned} \tag{A.13}$$

Now, for any multi-indices I, J and $(x, \xi, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times [1, \infty)$, we can write

$$\nabla_{x,I} \nabla_{\xi,J} r_L(x, \xi, \lambda) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-iz\cdot\xi} K_{I,J}(x, z, \lambda)\, dz, \tag{A.14}$$
$$K_{I,J}(x, z, \lambda) = (-iz)_J \sum_{I'+I''=I} \nabla_{x,I'} \nabla_{y,I''} K_\lambda(x, x - z).$$

(Note the above is justified, as the integral in (A.14) is absolutely convergent by (A.9).) By the same integration by parts trick as in the proof of Proposition A.13, we then bound

$$\begin{aligned} |\nabla_{x,I} \nabla_{\xi,J} r_L(x, \xi, \lambda)| &\leq C'(\lambda^2 + |\xi|^2)^{-M} \int_{\mathbb{R}^n} |(\lambda^2 + \Delta_z)^M K_{I,J}(x, z, \lambda)|\, dz \\ &\leq C'(\lambda^2 + |\xi|^2)^{-M}, \end{aligned}$$

for any $M \geq n$, where in the last step, we used (A.14) along with the infinite-order decay from (A.9). In particular, this implies $r_L \in \Lambda^s_\delta(\mathbb{R}^n)$ for every $s \in \mathbb{R}$; an analogous process starting from the second identity in (A.13) yields $r_R \in \Lambda^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$.

Finally, a direct computation yields, for any $\psi \in C^\infty_0(\mathbb{R}^n; \mathbb{C})$ and $x \in \mathbb{R}^n$,

$$\begin{aligned} \mathrm{op}^L_\lambda(r_L)\psi(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} e^{-iz\cdot\xi} K_\lambda(x, x - z)\psi(y)\, d\xi dy dz \\ &= \int_{\mathbb{R}^n} \mathcal{F}^{-1}[K_\lambda(x, \cdot)](\xi)\mathcal{F}[\psi](\xi)\, d\xi, \end{aligned}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier and inverse Fourier transforms, respectively. Using the adjoint properties of $\mathcal{F}$ and the Fourier inversion formula, we have, from (A.8),

$$\begin{aligned} \mathrm{op}^L_\lambda(r_L)\psi(x) &= \int_{\mathbb{R}^n} K_\lambda(x, y)\psi(y)\, dy \\ &= A_\lambda\psi(x). \end{aligned}$$

To complete the proof of (A.12), we apply a similar computation using (A.4):

$$\begin{aligned} \mathrm{op}^R_\lambda(r_R)\psi(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} e^{-iz\cdot\xi} K_\lambda(y + z, y)\psi(y)\, d\xi dy dz \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \mathcal{F}[K_\lambda(\cdot, y)](\xi)\psi(y)\, d\xi dy \\ &= \int_{\mathbb{R}^n} K_\lambda(x, y)\psi(y)\, d\xi \\ &= A_\lambda\psi(x). \qquad \square \end{aligned}$$

Proposition A.16. *Given $s \in \mathbb{R}$, $\delta \in [0, 1)$, and $b \in \bar{\Lambda}^s_\delta(\mathbb{R}^n)$, there exist $b_L, b_R \in \Lambda^s_\delta(\mathbb{R}^n)$ such that the following identities hold for all $\lambda \in [1, \infty)$:*

$$\mathrm{Op}_\lambda(b) = \mathrm{op}^L_\lambda(b_L) = \mathrm{op}^R_\lambda(b_R), \qquad \lambda \in [1, \infty). \tag{A.15}$$

Moreover, there exist $b_{L,r}, b_{R,r} \in \Lambda^{s-2(1-\delta)}_\delta(\mathbb{R}^n)$ such that for any $x, y, \xi \in \mathbb{R}^n$ and $\lambda \in [1, \infty)$,

$$b_L(x, \xi, \lambda) = b(x, x, \xi, \lambda) - i(\nabla_\xi \cdot \nabla_y b)(x, x, \xi, \lambda) + b_{L,r}(x, \xi, \lambda), \tag{A.16}$$

$$b_R(y,\xi,\lambda) = b(y,y,\xi,\lambda) + i(\nabla_\xi \cdot \nabla_x b)(y,y,\xi,\lambda) + b_{R,r}(y,\xi,\lambda).$$

Proof. First, by Taylor's theorem, we can write

$$\begin{aligned} b(x,y,\xi,\lambda) &= \sum_{|I|<N} \tfrac{(y-x)_I \nabla_{y,I} b(x,x,\xi,\lambda)}{k!} + \sum_{|I|=N} \tfrac{(y-x)_I \int_0^1 \nabla_{y,I} b(x,(1-t)x+ty,\xi,\lambda)\,(1-t)^{N-1}\,dt}{(N-1)!}, \\ b(x,y,\xi,\lambda) &= \sum_{|I|<N} \tfrac{(x-y)_I \nabla_{x,I} b(y,y,\xi,\lambda)}{k!} + \sum_{|I|=N} \tfrac{(x-y)_I \int_0^1 \nabla_{x,I} b(tx+(1-t)y,y,\xi,\lambda)\,(1-t)^{N-1}\,dt}{(N-1)!}. \end{aligned} \tag{A.17}$$

for any $x, y, \xi \in \mathbb{R}^n$, $\lambda \in [1,\infty)$, and $N \in \mathbb{N}$. Recalling (A.6), noting that

$$(x-y)e^{i(x-y)\cdot\xi} = -i\nabla_\xi e^{i\,(x-y)\cdot\xi}$$

and integrating by parts, we obtain, for any x, λ as above and $\psi \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$,

$$\begin{aligned} \mathrm{Op}_\lambda(b)\psi(x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\left[\sum_{k<N} b_{L,k}(x,\xi,\lambda) + r_{L,N}(x,y,\xi,\lambda)\right]\psi(y)\,dyd\xi, \\ \mathrm{Op}_\lambda(b)\psi(x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\left[\sum_{k<N} b_{R,k}(y,\xi,\lambda) + r_{R,N}(x,y,\xi,\lambda)\right]\psi(y)\,dyd\xi, \end{aligned} \tag{A.18}$$

where we have, for each $N \in \mathbb{N}$ and $0 \leq k < N$,

$$\begin{aligned} b_{L,k} &= \tfrac{1}{k!}\,(-i)^k(\nabla_\xi\cdot\nabla_y)^k b(x,x,\xi,\lambda) \in \Lambda_\delta^{s-k(1-\delta)}(\mathbb{R}^n), \\ b_{R,k} &= \tfrac{1}{k!}\,(+i)^k(\nabla_\xi\cdot\nabla_x)^k b(y,y,\xi,\lambda) \in \Lambda_\delta^{s-k(1-\delta)}(\mathbb{R}^n), \\ r_{L,N} &= \tfrac{(-i)^N}{(N-1)!}\int_0^1 (\nabla_\xi\cdot\nabla_y)^N b(x,(1-t)x+ty,\xi,\lambda)\,(1-t)^{N-1}\,dt \in \bar{\Lambda}_\delta^{s-N(1-\delta)}(\mathbb{R}^n), \\ r_{R,N} &= \tfrac{(+i)^N}{(N-1)!}\int_0^1 (\nabla_\xi\cdot\nabla_x)^N b(tx+(1-t)y,y,\xi,\lambda)\,(1-t)^{N-1}\,dt \in \bar{\Lambda}_\delta^{s-N(1-\delta)}(\mathbb{R}^n). \end{aligned} \tag{A.19}$$

We now define $b_{L,*} : \mathbb{R}^n \times \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ and $r_{L,*} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ by

$$\begin{aligned} b_{L,*}(x,\xi,\lambda) &:= \sum_{k=0}^\infty \chi(\epsilon_k\xi) b_{L,k}(x,\xi,\lambda), \\ r_{L,*}(x,y,\xi,\lambda) &:= b(x,y,\xi,\lambda) - b_{L,*}(x,\xi,\lambda), \end{aligned} \tag{A.20}$$

where $\chi \in C^\infty(\mathbb{R}^n;[0,1])$ is chosen to satisfy

$$\chi|_{\{\xi\in\mathbb{R}^n \,|\, |\xi|\leq 1\}} \equiv 0, \qquad \chi|_{\{\xi\in\mathbb{R}^n \,|\, |\xi|\geq 2\}} \equiv 1, \tag{A.21}$$

and where the decreasing sequence $(\epsilon_k)_{k\geq 0}$ is chosen to converge to zero fast enough so that

$$\sum_{|I|+|J|\leq k} |\nabla_{x,I}\nabla_{\xi,J} b_{L,k}(x,\xi,\lambda)| \leq 2^{-k}\gamma(\xi,\lambda)^{s-|J|+\delta|I|+1}, \qquad k \geq 0, \quad |\xi| \geq \epsilon_k^{-1}. \tag{A.22}$$

In particular, the sum in (A.20) converges, since it is finite at each $\xi \in \mathbb{R}^n$. Moreover,

$$\sum_{k=N}^\infty \chi(\epsilon_k\xi) b_{L,k}(x,\xi,\lambda) \in \Lambda_\delta^{s-N(1-\delta)+1}(\mathbb{R}^n), \qquad N > 0, \tag{A.23}$$

since (A.22) also implies all derivatives of the sum in (A.20) converge uniformly. (In fact, the quantity in (A.23) also lies in $\Lambda_\delta^{s-N(1-\delta)}(\mathbb{R}^n)$; this can be seen by applying (A.23) with N replaced by $N' \gg N$ and noting that each $b_{L,k}$ itself lies in $\Lambda_\delta^{s-k(1-\delta)}(\mathbb{R}^n)$.)

As a result of (A.18)–(A.20) and (A.23), we can write, for each $N > 0$ and $\lambda \in [1, \infty)$,

$$\begin{aligned} \mathrm{Op}_\lambda(r_{L,*}) &= \mathrm{Op}_\lambda(r_{L,N}) + \mathrm{op}^L_\lambda(r^1_{L,N}) + \mathrm{op}^L_\lambda(r^2_{L,N}), \\ r^1_{L,N}(x,\xi,\lambda) &= \sum_{k<N} [1 - \chi(\epsilon_k \xi)] b_{L,k}(x,\xi,\lambda), \\ r^2_{L,N}(x,\xi,\lambda) &= \sum_{k\geq N} \chi(\epsilon_k \xi) b_{L,k}(x,\xi,\lambda). \end{aligned} \tag{A.24}$$

Note $r^1_{L,N} \in \bigcap_{t\in\mathbb{R}} \Lambda^t_\delta(\mathbb{R}^n)$, since it is uniformly compactly supported in ξ. Thus, by (A.19) and (A.23), we have $r_{L,*} \in \bar{\Lambda}_\delta^{s-N(1-\delta)}(\mathbb{R}^n)$ for all $N > 0$, hence $(\mathrm{Op}_\lambda(r_{L,*}))_{\lambda\in[1,\infty)}$ is a δ-residual family of operators. Thus, Lemma A.15 yields an $r_L \in \bigcap_{t\in\mathbb{R}} \Lambda^t_\delta(\mathbb{R}^n)$ with

$$\mathrm{Op}_\lambda(r_{L,*}) = \mathrm{op}^L_\lambda(r_L), \qquad \lambda \in [1, \infty),$$

and hence it follows that

$$b_L := b_{L,*} + r_{L,*} \in \Lambda^s_\delta(\mathbb{R}^n)$$

satisfies the first part of (A.15); an analogous process also yields $b_R \in \Lambda^s_\delta(\mathbb{R}^n)$.

Finally, the identities (A.16) follow by expanding (A.18) for large enough N and recalling (A.23) (for b_L) and its analogue for right quantification. □

Remark A.17. *Note the expansions* (A.16) *can be taken to higher orders as in* (A.18) *and* (A.19)*. However, we will not require such formulas in these notes.*

A.3. **Operator Properties.** The aim of this subsection to prove generalizations of Propositions 3.6 and 3.10, which were left untreated in earlier sections. In particular, Propositions 3.6 and 3.10 will follow immediately from the following two results:

Proposition A.18. *Let $s, s_1, s_2 \in \mathbb{R}$ and $\delta \in [0, 1)$. Then:*

- *Given $b_1 \in \Lambda^{s_1}_\delta(\mathbb{R}^n)$ and $b_2 \in \Lambda^{s_2}_\delta(\mathbb{R}^n)$, there exists $r \in \Lambda_\delta^{s_1+s_2-2(1-\delta)}(\mathbb{R}^n)$ such that*

$$\mathrm{op}^L_\lambda(b_1)\,\mathrm{op}^L_\lambda(b_2) = \mathrm{op}^L_\lambda(b_1 b_2 - i\nabla_\xi b_1 \cdot \nabla_x b_2 + r), \qquad \lambda \in [1, \infty). \tag{A.25}$$

- *Given $b_1 \in \Lambda^{s_1}_\delta(\mathbb{R}^n)$ and $b_2 \in \Lambda^{s_2}_\delta(\mathbb{R}^n)$, there exists $r \in \Lambda_\delta^{s_1+s_2-2(1-\delta)}(\mathbb{R}^n)$ such that*

$$[\mathrm{op}^L_\lambda(b_1), \mathrm{op}^L_\lambda(b_2)] = \mathrm{op}^L_\lambda(-i\{b_1, b_2\} + r), \qquad \lambda \in [1, \infty). \tag{A.26}$$

- *Given $b \in \Lambda^s_\delta(\mathbb{R}^n)$, there exists $r \in \Lambda_\delta^{s-(1-\delta)}(\mathbb{R}^n)$ such that*

$$\mathrm{op}^L_\lambda(b)^* = \mathrm{op}^L_\lambda(\bar{b} + r), \qquad \lambda \in [1, \infty). \tag{A.27}$$

Proof. We fix $\lambda \in [1, \infty)$ throughout the proof. Given $\psi, \varphi \in C^\infty_0(\mathbb{R}^n; \mathbb{C})$, we have

$$\begin{aligned} \langle \psi, \mathrm{op}^L_\lambda(b)\varphi \rangle_{L^2(\mathbb{R}^n)} &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \psi(x) \overline{e^{i(x-y)\cdot\xi} b(x,\xi,\lambda)\varphi(y)}\, d\xi dy dx \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\, \overline{b(y,\xi,\lambda)} \psi(y) \overline{\varphi(x)}\, d\xi dy dx, \end{aligned}$$

and it hence follows that

$$\mathrm{op}^L_\lambda(b)^* = \mathrm{op}^R_\lambda(\bar{b}).$$

Viewing $\bar{b}$ in the above as an x-independent element of $\bar{\Lambda}^s_\delta(\mathbb{R}^n)$, then by Proposition A.16,

$$\mathrm{op}^L_\lambda(b)^* = \mathrm{op}^L_\lambda(\bar{b} - i\nabla_\xi \cdot \nabla_x \bar{b} + r')$$

for some $r' \in \Lambda^{s-2(1-\delta)}_\delta(\mathbb{R}^n)$, which in particular proves (A.27).

Next, for (A.25), by Proposition A.16, there exists $b_{2,r} \in \Lambda^{s_2}_\delta(\mathbb{R}^n)$ such that

$$\text{(A.28)} \qquad \mathrm{op}^L_\lambda(b_2) = \mathrm{op}^R_\lambda(b_{2,r}), \qquad b_{2,r} = b_2 + i(\nabla_\xi \cdot \nabla_x b_2) + r_2, \qquad r_2 \in \Lambda^{s_2-2(1-\delta)}_\delta(\mathbb{R}^n).$$

(For the expansion in (A.28), we viewed b_2 as a y-independent element of $\bar{\Lambda}^{s_2}_\delta(\mathbb{R}^n)$ and applied (A.16).) By (A.3) and (A.4), we have, for any $\psi \in C^\infty_0(\mathbb{R}^n;\mathbb{C})$ and $x \in \mathbb{R}^n$,

$$\text{(A.29)} \qquad \begin{aligned} \mathrm{op}^L_\lambda(b_1)\,\mathrm{op}^L_\lambda(b_2)\psi(x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} b_1(x,\xi,\lambda)\mathcal{F}[\mathrm{op}^R_\lambda(b_{2,r})\psi](\xi)\,d\xi \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\,x\cdot\xi} b_1(x,\xi,\lambda)\int_{\mathbb{R}^n} e^{-i\,y\cdot\xi} b_{2,r}(y,\xi,\lambda)\psi(y)\,dy\,d\xi \\ &= \mathrm{Op}_\lambda(b_3), \end{aligned}$$

where $b_3 \in \bar{\Lambda}^{s_1+s_2}_\delta(\mathbb{R}^n)$ is given by

$$\begin{aligned} b_3(x,y,\xi,\lambda) &= b_1(x,\xi,\lambda)b_{2,r}(y,\xi,\lambda) \\ &= b_1(x,\xi,\lambda)(b_2 + i\nabla_\xi \cdot \nabla_x b_2 + r_2)(y,\xi,\lambda). \end{aligned}$$

Applying Proposition A.16 to b_3 above, we see there exists $b_4 \in \Lambda^{s_1+s_2}_\delta(\mathbb{R}^n)$ with

$$\text{(A.30)} \qquad \begin{aligned} \mathrm{Op}_\lambda(b_3) &= \mathrm{op}^L_\lambda(b_4), \\ b_4(x,\xi,\lambda) &= b_3(x,x,\xi,\lambda) - i(\nabla_\xi \cdot \nabla_y b_3)(x,x,\xi,\lambda) + r_4(x,\xi,\lambda) \\ &= b_1(x,\xi,\lambda)b_2(x,\xi,\lambda) - i\,\nabla_\xi b_1(x,\xi,\lambda)\cdot\nabla_x b_2(x,\xi,\lambda) \\ &\qquad - i\nabla_\xi \cdot [b_1 \nabla_x(i\nabla_\xi\cdot\nabla_x b_2 + r_2)](x,\xi,\lambda) + r_4(x,\xi,\lambda), \end{aligned}$$

for any $x, \xi \in \mathbb{R}^n$, and for some $r_4 \in \Lambda^{s_1+s_2-2(1-\delta)}_\delta(\mathbb{R}^n)$. In particular, (A.30) implies

$$b_4 = b_1 b_2 - i\nabla_\xi b_1 \cdot \nabla_x b_2 + r_0, \qquad r_0 \in \Lambda^{s_1+s_2-2(1-\delta)}_\delta(\mathbb{R}^n).$$

Combining (A.29)–(A.30) and the above results in (A.25).

Finally, (A.26) follows immediately by noting that

$$[\mathrm{op}^L_\lambda(b_1), \mathrm{op}^L_\lambda(b_2)] = \mathrm{op}^L_\lambda(b_1)\,\mathrm{op}^L_\lambda(b_2) - \mathrm{op}^L_\lambda(b_2)\,\mathrm{op}^L_\lambda(b_1),$$

and by then applying (A.25) twice. □

Corollary A.19. *Let $s, s_1, s_2 \in \mathbb{R}$ and $\delta \in [0,1)$. Moreover, let $b \in \bar{\Lambda}^s_\delta(\mathbb{R}^n)$, $b_1 \in \bar{\Lambda}^{s_1}_\delta(\mathbb{R}^n)$, $b_2 \in \bar{\Lambda}^{s_2}_\delta(\mathbb{R}^n)$. Then, there exist $b_c \in \bar{\Lambda}^{s_1+s_2}_\delta(\mathbb{R}^n)$ and $b_* \in \bar{\Lambda}^s_\delta(\mathbb{R}^n)$ such that*

$$\text{(A.31)} \qquad \mathrm{Op}_\lambda(b_1)\,\mathrm{Op}_\lambda(b_2) = \mathrm{Op}_\lambda(b_c), \qquad \mathrm{Op}_\lambda(b)^* = \mathrm{Op}_\lambda(b_*), \qquad \lambda \in [1,\infty).$$

Proof. This is a direct consequence of Propositions A.16 and A.18. □

Proposition A.20. *Let $s \in \mathbb{R}$, $\delta \in [0,1)$, and $b \in \bar{\Lambda}^s_\delta(\mathbb{R}^n)$. Then, there exists $C > 0$ with*

$$\| \operatorname{Op}_\lambda(b)\psi\|_{L^2(\mathbb{R}^n)} \leq C\|\psi\|_{\mathcal{H}^s_\lambda(\mathbb{R}^n)}, \qquad \psi \in C_0^\infty(\mathbb{R}^n;\mathbb{C}), \quad \lambda \in [1,\infty). \tag{A.32}$$

Similarly, there exists $C > 0$ such that

$$\langle \operatorname{Op}_\lambda(b)\psi, \psi\rangle_{L^2(\mathbb{R}^n)} \leq C\|\psi\|_{\mathcal{H}^{\frac{s}{2}}_\lambda(\mathbb{R}^n)}, \qquad \psi \in C_0^\infty(\mathbb{R}^n;\mathbb{C}), \quad \lambda \in [1,\infty). \tag{A.33}$$

Proof. First, we claim that if $s < -n$, then

$$\| \operatorname{Op}_\lambda(b)\psi\|_{L^2(\mathbb{R}^n)} \leq C\|\psi\|_{L^2(\mathbb{R}^n)}, \qquad \psi \in C_0^\infty(\mathbb{R}^n;\mathbb{C}), \quad \lambda \in [1,\infty). \tag{A.34}$$

To prove this, we define K_λ by the formula (A.10), with b as in the proposition statement. Using this, we then apply a direct computation to obtain, for any $\varphi \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$,

$$\begin{aligned} |\langle \operatorname{Op}_\lambda(b)\psi, \varphi\rangle_{L^2(\mathbb{R}^n)}|^2 &= \frac{1}{(2\pi)^{2n}} \left| \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K_\lambda(x,y)\psi(y)\overline{\varphi(x)}\, dydx \right|^2 \\ &\leq C' \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |K_\lambda(x,y)||\psi(y)|^2\, dydx \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |K_\lambda(x,y)||\varphi(x)|^2\, dydx \\ &\leq C'\|\psi\|_{L^2(\mathbb{R}^n)}\|\varphi\|_{L^2(\mathbb{R}^n)}, \end{aligned}$$

for some constants $C' > 0$ (which can change between lines) that are independent of λ. To obtain the last step above, we follow the proof of Proposition A.13 to again derive

$$|K_\lambda(x,y)| \leq C'(1 + |x-y|^2)^{-M}, \qquad x, y \in \mathbb{R}^n, \quad M \geq 0,$$

for $C' > 0$ as above. This proves of the claim (A.34) for $s < -n$.

Next, we claim that (A.34) holds for any $s \leq -(1-\delta)$. For this, note that if $s < -\frac{n}{2}$, then

$$\| \operatorname{Op}_\lambda(b)\psi\|^2_{L^2(\mathbb{R}^n)} = \langle \operatorname{Op}_\lambda(b)^* \operatorname{Op}_\lambda(b)\psi, \psi\rangle_{L^2(\mathbb{R}^n)}. \tag{A.35}$$

Moreover, by Corollary A.19, we see that

$$\operatorname{Op}_\lambda(b)^* \operatorname{Op}_\lambda(b)\psi = \operatorname{Op}_\lambda(b_2), \qquad b_2 \in \bar{\Lambda}^{2s}_\delta(\mathbb{R}^n).$$

Thus, (A.34) follows from the previous claim and (A.35), since $2s < -n$. Lastly, for general $s \leq -(1-\delta)$, we iterate the above process a fixed number of times (noting that each such iteration doubles the values of s allowed) to again derive (A.34).

We now claim that (A.34) holds when $s = 0$. In this case, (A.2) implies that b is uniformly bounded, in particular over all $\lambda \in [1,\infty)$. As a result, we fix a constant

$$\mathcal{C} > \sup_{(x,y,\xi,\lambda)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times[1,\infty)} |b(x,\xi,\lambda)|^2, \tag{A.36}$$

and we define the quantities (for each $\lambda \in [1,\infty)$ in the latter case)

$$b_0 := \sqrt{\mathcal{C} - |b|^2} \in \bar{\Lambda}^0_\delta(\mathbb{R}^n), \qquad R_\lambda := \mathcal{C} - \operatorname{Op}_\lambda(b)^* \operatorname{Op}_\lambda(b) - \operatorname{Op}_\lambda(b_0)^* \operatorname{Op}_\lambda(b_0). \tag{A.37}$$

Proposition A.18 and Corollary A.19 together yield some $r \in \bar{\Lambda}^{-(1-\delta)}_\delta(\mathbb{R}^n)$ with

$$\begin{aligned} R_\lambda &= \operatorname{Op}_\lambda(\mathcal{C} - |b|^2 - b_0^2 + r) \\ &= \operatorname{Op}_\lambda(r), \end{aligned} \tag{A.38}$$

for every $\lambda \in [1, \infty)$. Finally, applying (A.37), (A.38), and the preceding claim, we obtain

$$\begin{aligned} \| \operatorname{Op}_\lambda(b)\psi\|^2_{L^2(\mathbb{R}^n)} &= \langle \operatorname{Op}_\lambda(b)^* \operatorname{Op}_\lambda(b)\psi, \psi\rangle_{L^2(\mathbb{R}^n)} \\ &= \mathcal{C}\langle \psi, \psi\rangle_{L^2(\mathbb{R}^n)} - \langle \operatorname{Op}_\lambda(b_0)\psi, \operatorname{Op}_\lambda(b_0)\psi\rangle_{L^2(\mathbb{R}^n)} - \langle \operatorname{Op}_\lambda(r)\psi, \psi\rangle_{L^2(\mathbb{R}^n)} \\ &\leq \mathcal{C}\|\psi\|^2_{L^2(\mathbb{R}^n)} + \| \operatorname{Op}_\lambda(r)\psi\|_{L^2(\mathbb{R}^n)}\|\psi\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Our desired claim now follows from the above and the preceding claim for $s \leq -(1-\delta)$.

In particular, the above claim yields (A.32) for $s = 0$. To prove (A.32) for general $s \in \mathbb{R}$, we note, using (3.7) and Corollary A.19, that

$$\operatorname{Op}_\lambda(b)(\lambda^2 - \Delta)^{-\frac{s}{2}} = \operatorname{Op}_\lambda(b_0),$$

for some $b_0 \in \bar{\Lambda}^0_\delta(\mathbb{R}^n)$. Thus, applying the already established $s = 0$ case yields

$$\begin{aligned} \| \operatorname{Op}_\lambda(b)\psi\|_{L^2(\mathbb{R}^n)} &\leq C\| \operatorname{Op}_\lambda(b_0)(\lambda^2 - \Delta)^{\frac{s}{2}}\psi\|_{L^2(\mathbb{R}^n)} \\ &\leq C\|\psi\|_{\mathcal{H}^s_\lambda(\mathbb{R}^n)}, \end{aligned}$$

which completes the proof of (A.32). Similarly, for (A.33), we bound

$$\begin{aligned} \langle \operatorname{Op}_\lambda(b)\psi, \psi\rangle_{L^2(\mathbb{R}^n)} &= \|(\lambda^2 - \Delta)^{-\frac{s}{4}} \operatorname{Op}_\lambda(b)\psi\|_{L^2(\mathbb{R}^n)}\|(\lambda^2 - \Delta)^{\frac{s}{4}}\psi\|_{L^2(\mathbb{R}^n)} \\ &\leq \|\psi\|^2_{\mathcal{H}^{\frac{s}{2}}_\lambda(\mathbb{R}^n)}, \end{aligned}$$

where we applied (A.32) in the final step. □

A.4. **The Sharp Gårding Inequality.** The final task of the appendix is to prove the sharp Gårding inequality of Theorem 3.11, which we now recall in our current language:

Theorem A.21. *Let $s \in \mathbb{R}$ and $\lambda_0 \geq 1$, and suppose $b \in \Lambda^s_0(\mathbb{R}^n)$ satisfies*

$$b(x, \xi, \lambda) \geq 0, \qquad (x, \xi, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times [\lambda_0, \infty). \tag{A.39}$$

Then, there exists $C > 0$ such that for any $\lambda \geq \lambda_0$ and $\psi \in C^\infty_0(\mathbb{R}^n; \mathbb{C})$,

$$\operatorname{Re}\langle \operatorname{op}^L_\lambda(b)\psi, \psi\rangle_{L^2(\mathbb{R}^n)} \geq -C\|\psi\|^2_{\mathcal{H}_\lambda^{\frac{s-1}{2}}(\mathbb{R}^n)}. \tag{A.40}$$

Proof. Let ψ and λ be as in the theorem statement. First, by (A.27), we have

$$\begin{aligned} 2\operatorname{Re}\langle \operatorname{op}^L_\lambda(i \operatorname{Im} b)\psi, \psi\rangle_{L^2(\mathbb{R}^n)} &= \operatorname{Re}\langle [\operatorname{op}^L_\lambda(i \operatorname{Im} b) + \operatorname{op}^L_\lambda(i \operatorname{Im} b)^*]\psi, \psi\rangle_{L^2(\mathbb{R}^n)} \\ &= \operatorname{Re}\langle \operatorname{op}^L_\lambda(r)\psi, \psi\rangle_{L^2(\mathbb{R}^n)}, \end{aligned}$$

for some $r \in \Lambda^{s-1}_0(\mathbb{R}^n)$. Thus, applying (A.33) results in the inequality

$$\operatorname{Re}\langle \operatorname{op}_\lambda(i \operatorname{Im} b)\psi, \psi\rangle_{L^2(\mathbb{R}^n)} \geq -C\|\psi\|^2_{\mathcal{H}_\lambda^{\frac{s-1}{2}}(\mathbb{R}^n)},$$

with the constant independent of λ. Therefore, in the remainder of the proof, it suffices to write b in place of $\operatorname{Re} b$, that is, to assume b is purely real-valued.

Fix now a real-valued function $\phi \in C^\infty_0(\mathbb{R}^n; \mathbb{R})$ satisfying

$$\|\phi\|_{L^2(\mathbb{R}^n)} = 1, \qquad \phi(-x) = \phi(x), \quad x \in \mathbb{R}^n. \tag{A.41}$$

We now define $b^\star \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times (0,\infty);\mathbb{C})$ by

$$b^\star(x,y,\xi,\lambda) := \gamma(\xi,\lambda)^{\frac{n}{2}} \int_{\mathbb{R}^n} \phi\big(\gamma(\xi,\lambda)^{\frac{1}{2}}(x-z)\big)\phi\big(\gamma(\xi,\lambda)^{\frac{1}{2}}(y-z)\big)\, b(z,\lambda,\xi)\, dz, \tag{A.42}$$

with γ defined as in (A.1). An inspection of (A.42) yields that $b^\star \in \bar{\Lambda}^s_{\frac{1}{2}}(\mathbb{R}^n)$. Moreover, by Proposition A.16, we can find $b^\star_L \in \Lambda^s_{\frac{1}{2}}(\mathbb{R}^n)$ such that

$$\mathrm{Op}_\lambda(b^\star) = \mathrm{op}^L_\lambda(b^\star_L). \tag{A.43}$$

Now, a direct computation using (A.3), (A.39), (A.42), and (A.43) then gives

$$\begin{aligned} \mathrm{Re}\langle \mathrm{op}^L_\lambda(b^\star_L)\psi,\psi\rangle_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \gamma(\xi,\lambda)^{\frac{n}{2}} \left| \int_{\mathbb{R}^n} \phi\big(\gamma(\xi,\lambda)^{\frac{1}{2}}(x-z)\big)\psi(x)\,dx\right|^2 b(z,\xi,\lambda)\,dzd\xi \\ &\geq 0, \end{aligned}$$

as long as $\lambda \geq \lambda_0$. As a result, we obtain

$$\mathrm{Re}\langle \mathrm{op}^L_\lambda(b)\psi,\psi\rangle_{L^2(\mathbb{R}^n)} \geq \mathrm{Re}\langle \mathrm{op}^L_\lambda(b-b^\star_L)\psi,\psi\rangle_{L^2(\mathbb{R}^n)}, \qquad \lambda \geq \lambda_0. \tag{A.44}$$

Now, by Proposition A.16, we can write, for any $(x,y) \in \mathbb{R}^n$ and λ as before,

$$b^*_L(x,\xi,\lambda) = b^*(x,x,\xi,\lambda) - i(\nabla_\xi\cdot\nabla_y)b^*(x,x,\xi,\lambda) + r^*_L(x,\xi,\lambda), \tag{A.45}$$

with $r_L \in \Lambda^{s-1}_{\frac{1}{2}}(\mathbb{R}^n)$. Further, by a change of variables, we then compute

$$\begin{aligned} b^*(x,x,\xi,\lambda) &= \int_{\mathbb{R}^n} \phi(z)^2 b\big(x-\gamma(\xi,\lambda)^{-\frac{1}{2}}z,\xi,\lambda\big)\,dz, \qquad \text{(A.46)} \\ -i(\nabla_\xi\cdot\nabla_y)b^*(x,x,\xi,\lambda) &= -i\nabla_\xi\cdot\int_{\mathbb{R}^n}\gamma(\xi,\lambda)^{\frac{1}{2}}\phi(z)\nabla\phi(z)b\big(x-\gamma(\xi,\lambda)^{-\frac{1}{2}}z,\xi,\lambda\big)\,dz \\ &= -i\int_{\mathbb{R}^n}\phi(z)\nabla\phi(z)\cdot B\big(x-\gamma(\xi,\lambda)^{-\frac{1}{2}}z,\xi,\lambda\big)\,dz, \end{aligned}$$

with $B := (B_1,\dots,B_n)$ a vector-valued function such that $B_k \in \Lambda^{s-\frac{1}{2}}_0(\mathbb{R}^n)$, $1 \leq k \leq n$.

Applying Taylor's theorem, we can then expand

$$\begin{aligned} b^*(x,x,\xi,\lambda) &= b_{0,0}(x,\xi,\lambda) + b_{0,1}(x,\xi,\lambda) + b_{0,2}(x,\xi,\lambda), \qquad \text{(A.47)} \\ b_{0,0}(x,\xi,\lambda) &= b(x,\xi,\lambda)\int_{\mathbb{R}^n}\phi(z)^2\,dz, \\ b_{0,1}(x,\xi,\lambda) &= -\gamma(\xi,\lambda)^{-\frac{1}{2}}\nabla_x b(x,\xi,\lambda)\cdot\int_{\mathbb{R}^n}\phi(z)^2 z\,dz, \\ b_{0,2}(x,\xi,\lambda) &= \gamma(\xi,\lambda)^{-1}\int_{\mathbb{R}^n}\int_0^1(1-t)\phi(z)^2 z\cdot\nabla^2_x b\big(x-t\gamma(\xi,\lambda)^{-\frac{1}{2}}z,\xi,\lambda\big)\cdot z\,dtdz. \end{aligned}$$

Recalling (A.1), (A.2), and (A.41), while noting that $z \mapsto \phi^2(z)z$ is odd, then

$$b_{0,0}(x,\xi,\lambda) = b(x,\xi,\lambda), \qquad b_{0,1}(x,\xi,\lambda) = 0, \qquad b_{0,2} \in \Lambda^{s-1}_0(\mathbb{R}^n). \tag{A.48}$$

Similarly, applying Taylor's theorem again, we also expand

$$-i(\nabla_\xi\cdot\nabla_y)b^*(x,x,\xi,\lambda) = b_{1,0}(x,\xi,\lambda) + b_{1,1}(x,\xi,\lambda), \tag{A.49}$$

$$\begin{aligned} b_{1,0}(x,\xi,\lambda) &= -iB(x,\xi,\lambda) \cdot \int_{\mathbb{R}^n} \phi(z) \nabla \phi(z) \, dz \\ &= 0, \end{aligned} \tag{A.50}$$

since ϕ is compactly supported. Moreover, from the definition of B in (A.46), we obtain

$$b_{1,1} \in \Lambda_0^{s-1}(\mathbb{R}^n), \tag{A.51}$$

Finally, combining (A.45)–(A.51), we conclude that

$$b - b_L^\star \in \Lambda_{\frac{1}{2}}^{s-1}(\mathbb{R}^n).$$

The desired inequality (A.40) now follows from Proposition A.20, (A.44), and the above. □

Remark A.22. *Furthermore, if $b \in \Lambda_\delta^s(\mathbb{R}^n)$ instead in Theorem A.21, where $\delta \in [0,1)$, then* (A.40) *again holds, but with the Sobolev order $\frac{1}{2}(s-1)$ replaced by $\frac{1}{2}(s-(1-\delta))$.*

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