THE CHRIST-KISELEV LEMMA

ARICK SHAO

1. Introduction

The Christ-Kiselev lemma, as presented here, is a general boundedness property for certain integral transforms T involving a kernel. This version of the lemma states that, in certain spaces, if such an integral transform is bounded, then some restrictions of this integral transform to partial domains must also be bounded. This estimate has important applications in the study of dispersive partial differential equations, in particular in establishing Strichartz-type estimates. In this short note, we state and prove this property; the proof is based on that found in [3]. A more general version of this estimate can be found in the original paper, [1].

The most basic version of the integral Christ-Kiselev lemma is the following.

Theorem 1 (Christ, Kiselev)**.** *Consider a linear operator*

$$T : L^p(\mathbb{R}) \rightarrow L^q(\mathbb{R}), \qquad 1 \leq p < q < \infty,$$

such that T can be expressed as an integral transform

$$Tf(t) = \int_{\mathbb{R}} K(t,s) f(s) ds, \qquad K : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}.$$

If, for sufficiently nice $g : \mathbb{R} \rightarrow \mathbb{C}$, we define

$$\tilde{T}g(t) = \int_{-\infty}^{t} K(t,s) g(s) ds,$$

then $\tilde{T}$ extends to a bounded linear operator from $L^p(\mathbb{R})$ into $L^q(\mathbb{R})$.

This result can be directly generalized to the case in which our linear operator T (and in particular, the kernel K) acts on Banach spaces.

Theorem 2 (Christ, Kiselev)**.** *Let X and Y denote Banach spaces, and let*

$$T : L^p(\mathbb{R}; X) \rightarrow L^q(\mathbb{R}; Y), \qquad 1 \leq p < q < \infty,$$

such that T can be expressed as an integral transform

$$Tf(t) = \int_{\mathbb{R}} K(t,s) f(s) ds, \qquad K : \mathbb{R} \times \mathbb{R} \rightarrow B(X,Y),$$

with $B(X,Y)$ denoting the space of bounded linear transformations from X into Y. If, for sufficiently nice $g : \mathbb{R} \rightarrow X$, we define

$$\tilde{T}g(t) = \int_{-\infty}^{t} K(t,s) g(s) ds,$$

then $\tilde{T}$ extends to a bounded linear operator from $L^p(\mathbb{R}; X)$ into $L^q(\mathbb{R}; Y)$.

1.1. **Applications to Dispersive PDE.** In the study of dispersive partial differential equations, the Christ-Kiselev lemma plays an important role in the application of Strichartz estimates. To be more explicit, consider the initial value problem for the inhomogeneous linear Schrödinger equation,

$$i\partial_t u + \Delta u = F, \qquad u|_{t=0} = u_0. \tag{1}$$

Here, F and the unknown u are functions from $\mathbb{R} \times \mathbb{R}^d$ into $\mathbb{C}$, while the initial data u_0 is a function from $\mathbb{R}^d$ into $\mathbb{C}$. If u_0 and F are in sufficiently nice spaces, then by Duhamel's principle, the solution to (1) can be expressed as

$$u(t) = e^{it\Delta} u_0 + i \int_0^t e^{i(t-s)\Delta} F(s) ds. \tag{2}$$

Strichartz estimates can be used in conjunction with (2) to obtain spacetime bounds for u. For the homogeneous linear Schrödinger equation, one has

$$\| e^{it\Delta} u_0 \|_{L^q_t L^r_x (\mathbb{R} \times \mathbb{R}^d)} \lesssim_{d,q,r} \| u_0 \|_{L^2_x (\mathbb{R}^d)}, \tag{3}$$

where the parameters q, r, and d satisfy

$$q, r \in [2, \infty], \qquad \frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \qquad (q, r, d) \neq (2, \infty, 2).$$

This estimate controls the first term on the right-hand side of (2).

For the inhomogeneous term in (2), that is, the second term on the right-hand side, one must resort to the dual formulation of (3). Indeed, if

$$q', r' \in [1, 2], \qquad \frac{1}{q} + \frac{1}{q'} = 1, \qquad \frac{1}{r} + \frac{1}{r'} = 1,$$

then a standard duality argument applied to (3) yields

$$\left\| \int_{\mathbb{R}} e^{-is\Delta} F(s) ds \right\|_{L^2_x (\mathbb{R}^d)} \lesssim_{d,q,r} \| F \|_{L^{q'}_t L^{r'}_x (\mathbb{R} \times \mathbb{R}^d)}. \tag{4}$$

Furthermore, combining (3) and (4), we obtain the estimate

$$\left\| \int_{\mathbb{R}} e^{i(t-s)\Delta} F(s) ds \right\|_{L^q_t L^r_x (\mathbb{R} \times \mathbb{R}^d)} \lesssim_{d,q,r} \| F \|_{L^{q'}_t L^{r'}_x (\mathbb{R} \times \mathbb{R}^d)}. \tag{5}$$

However, (5) still does not quite apply to (2). At a fixed time t, the time integral in the left-hand side of (5) is over all of $\mathbb{R}$, while the corresponding integral on the right-hand side in (2) only goes up to t. Consequently, in order to convert (5) into an applicable estimate, we must apply the Christ-Kiselev lemma, i.e., Theorem 2. Finally, by combining all the above, we obtain the following bound for u:

$$\| u \|_{L^q_t L^r_x ([0, \infty) \times \mathbb{R}^d)} \lesssim \| u_0 \|_{L^2_x (\mathbb{R}^d)} + \| F \|_{L^{q'}_t L^{r'}_x ([0, \infty) \times \mathbb{R}^d)}. \tag{6}$$

A similar argument can, of course, be made for negative times.

For further background regarding dispersive partial differential equations and Strichartz estimates, the reader is referred to [4].

2. Proof of Theorem 2

First, without loss of generality, we can assume that

$$\|f\|^p_{L^p(\mathbb{R};X)} = \int_{\mathbb{R}} \|f(s)\|^p_X ds = 1.$$

Thus, our goal is to show that

$$\|\tilde{T}f\|_{L^q(\mathbb{R};Y)} \lesssim 1.$$

2.1. **The Whitney Decomposition.**

We first define some basic terminology. Recall that the *dyadic intervals* are subsets of $\mathbb{R}$ of the form

$$I_{j,k} = (k2^{-j}, (k+1)2^{-j}], \qquad j, k \in \mathbb{Z}.$$

Moreover, a *dyadic square* is a subset of $\mathbb{R}^2$ of the form

$$Q_{j,k,l} = I_{j,k} \times I_{j,l}, \qquad j, k, l \in \mathbb{Z}.$$

For any dyadic square Q, we define $\pi_1(Q)$ and $\pi_2(Q)$ to be the projections of Q to its first and second components, respectively. In other words,

$$\pi_1(Q_{j,k,l}) = I_{j,k}, \qquad \pi_2(Q_{j,k,l}) = I_{j,l}.$$

Furthermore, given Q as above, we let $l(Q)$ denote the side length of Q, e.g.,

$$l(Q_{j,k,l}) = 2^{-j}.$$

The *Whitney covering lemma* states the following.

Lemma 3 (Whitney Covering Lemma)**.** *Let Ω be a proper open subset of $\mathbb{R}^2$. Then, Ω can be expressed as a disjoint union of (countably many) dyadic squares. Moreover, for any such dyadic square Q comprising Ω, we have that*

$$l(Q) < d(Q, \partial\Omega) \leq (1 + \sqrt{2}) \cdot l(Q),$$

where $d(Q, \partial\Omega)$ denotes the distance from Q to the boundary of Ω.

For the proof and a detailed discussion of the Whitney covering lemma, the reader is referred to [2]. The basic idea of the proof is as follows: for any $x \in \Omega$, we let Q_x be the largest dyadic square containing x such that

$$l(Q_x) < d(Q_x, \partial\Omega).$$

The maximality condition for Q_x implies that

$$l(Q_x) < d(Q_x, \partial\Omega) \leq (1 + \sqrt{2}) l(Q_x).$$

We add Q_x to our desired collection $\mathcal{D}$ of dyadic squares. To see that this produces a partition of Ω, we need only to observe that if $y \in Q_x$, then the corresponding dyadic square Q_y generated from y is equal to Q_x.

Remark. *The Whitney covering lemma extends naturally to open subsets of $\mathbb{R}^n$, with the dyadic squares replaced by n-dimensional dyadic cubes. In this text, however, we will only require the two-dimensional case.*

We now apply the Whitney covering lemma to the region

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid x < y\}.$$

This yields a collection $\mathcal{D}$ of pairwise disjoint dyadic squares, with

$$\Omega = \bigcup_{Q \in \mathcal{D}} Q.$$

We collect some basic observations about the decomposition $\mathcal{D}$.

Lemma 4. *The following properties hold:*

1. *For any* $0 < x < y < 1$*, there is a unique* $Q \in \mathcal{D}$ *such that* $(x, y) \in Q$*.*
2. *If* $Q \in \mathcal{D}$*, then* $\sup \pi_1(Q) < \inf \pi_2(Q)$*. In other words,* $\pi_2(Q)$ *lies to the right of* $\pi_1(Q)$*, and* $\pi_1(Q)$ *and* $\pi_2(Q)$ *are non-adjacent.*
3. *If* $(x, y) \in \Omega \cap ([0, 1] \times [0, 1])$*, and if* Q *is the dyadic square in* $\mathcal{D}$ *containing* (x, y)*, then* $l(Q) \leq 1$*, where* $l(Q)$ *is the side length of* Q*.*
4. *For any dyadic interval* J *satisfying* $J \cap [0, 1] \neq \emptyset$*, there exist at most four dyadic squares* $Q \in \mathcal{D}$ *such that* $\pi_2(Q) = J$*.*

Proof. Property (1) is immediate, since Ω is a disjoint union of the elements of $\mathcal{D}$. Moreover, (2) follows from the fact any $Q \in \mathcal{D}$ is contained in Ω and satisfies

$$d(Q, \partial\Omega) > 0.$$

For (3), we appeal to the observation

$$d(Q, \partial\Omega) \leq d((x, y), \partial\Omega) < 1,$$

and to our construction of $\mathcal{D}$, as described in the paragraph after Lemma 3.

Finally, for (4), suppose $Q_1, \ldots, Q_5 \in \mathcal{D}$ are distinct, with

$$I_i = \pi_1(Q_i), \qquad \pi_2(Q_i) = J, \qquad l = l(Q_i), \qquad i \in \{1, \ldots, 5\}.$$

Moreover, we can assume that these five intervals are ordered such that I_1 lies furthest to the left and I_5 lies furthest to the right. By our construction for $\mathcal{D}$, we have $l < d(Q_5, \partial\Omega)$. Let Q_1^* denote the parent square of Q_1, i.e., the unique dyadic square of side length $2l$ that contains Q_1. By the triangle inequality,

$$d(Q_1, \partial\Omega) \leq d(Q_1^*, \partial\Omega) + \sqrt{2} \cdot l.$$

Furthermore, if $\inf I_5 - \inf I_1 = D > 0$, then by geometric considerations,

$$d(Q_1, \partial\Omega) = d(Q_5, \partial\Omega) + \frac{D}{\sqrt{2}}.$$

Since the distance between $\inf I_1$ and $\inf I_5$ is at least $4l$, then

$$\begin{aligned} d(Q_1^*, \partial\Omega) &\geq d(Q_1, \partial\Omega) - \sqrt{2} \cdot l \\ &\geq d(Q_5, \partial\Omega) + \frac{4}{\sqrt{2}} \cdot l - \sqrt{2} \cdot l \\ &> l + \sqrt{2} \cdot l. \end{aligned}$$

As a result,

$$d(Q_1^*, \partial\Omega) > 2l = l(Q_1^*),$$

which, due to the maximality property defining $\mathcal{D}$, contradicts that $Q_1 \in \mathcal{D}$. $\square$

2.2. **Decomposition of $\tilde{T} f$.** The next step is to decompose $\tilde{T} f$, using the above Whitney decomposition of Ω. First of all, we define

$$F : \mathbb{R} \to [0, 1], \qquad F(t) = \int_{-\infty}^{t} \|f(s)\|_X^p ds,$$

which is a nondecreasing function onto $[0, 1]$.

Lemma 5. *If $t \in \mathbb{R}$, then for almost every $s \in \mathbb{R}$ with $s < t$,*

$$K(t,s)f(s) = \sum_{Q\in\mathcal{D}} \chi_{\pi_2(Q)}(F(t))K(t,s)[\chi_{\pi_1(Q)}(F(s))f(s)], \tag{7}$$

where χ_A denotes the characteristic functions over A.

Proof. For any $s < t$, we have two cases. First, if $F(s) < F(t)$, then by Lemma 4, there exists unique $Q^{s,t} \in \mathcal{D}$ such that $(F(s), F(t)) \in Q^{s,t}$. Thus, we have

$$\begin{aligned} K(t,s)f(s) &= \chi_{Q^{s,t}}(F(s),F(t)) \cdot K(t,s)f(s) \\ &= \chi_{\pi_2(Q^{s,t})}(F(t))K(t,s)[\chi_{\pi_1(Q^{s,t})}(F(s))f(s)] \\ &= \sum_{Q\in\mathcal{D}} \chi_{\pi_2(Q)}(F(t))K(t,s)[\chi_{\pi_1(Q)}(F(s))f(s)]. \end{aligned}$$

Next, note that by the definition of F,

$$\int_{F^{-1}(F(t))} \|f(\tau)\|_X^p \, d\tau = 0.$$

It follows that $f(s) = 0$ for almost all $s < t$ with $F(s) = F(t)$. Thus,

$$K(t,s)f(s) = 0 = \sum_{Q\in\mathcal{D}} \chi_{\pi_2(Q)}(F(t))K(t,s)[\chi_{\pi_1(Q)}(F(s))f(s)],$$

with the first equality holding for almost all such s, and with the second equality holding unconditionally. This completes the proof of the lemma. □

Applying Lemma 5, we can decompose $\tilde{T}f$ as follows:

$$\begin{aligned} \tilde{T}f(t) &= \sum_{Q\in\mathcal{D}} \int_{-\infty}^{t} \chi_{\pi_2(Q)}(F(t))K(t,s)[\chi_{\pi_1(Q)}(F(s))f(s)]ds \\ &= \sum_{Q\in\mathcal{D}} \chi_{\pi_2(Q)}(F(t)) \int_{\mathbb{R}} K(t,s)[\chi_{\pi_1(Q)}(F(s))f(s)]ds \\ &= \sum_{Q\in\mathcal{D}} \chi_{\pi_2(Q)}(F(t))T[(\chi_{\pi_1(Q)} \circ F) \cdot f](t). \end{aligned}$$

As the above holds for any $t \in \mathbb{R}$, we obtain the formula

$$\tilde{T}f = \sum_{Q\in\mathcal{D}} (\chi_{\pi_2(Q)} \circ F) \cdot T[(\chi_{\pi_1(Q)} \circ F) \cdot f]. \tag{8}$$

Finally, we refine (8) by further decomposing the sum:

$$\begin{aligned} \tilde{T}f &= \sum_{j=0}^{\infty} \sum_{\substack{Q\in\mathcal{D} \\ l(Q)=2^{-j}}} (\chi_{\pi_2(Q)} \circ F) \cdot T[(\chi_{\pi_1(Q)} \circ F) \cdot f] \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} \sum_{\substack{Q\in\mathcal{D} \\ \pi_2(Q)=I_{j,k}}} (\chi_{I_{j,k}} \circ F) \cdot T[(\chi_{\pi_1(Q)} \circ F) \cdot f]. \end{aligned} \tag{9}$$

In particular, the terms in the above in which $\pi_2(Q)$ is either $I_{j,-1}$ or $I_{j,2^j}$ can be discarded, as all these terms vanish by the definition of F.

2.3. **Estimating $\tilde{T}f$.** From (9), we immediately estimate

$$\|\tilde{T}f\|_{L^q(\mathbb{R};Y)} \leq \sum_{j=0}^{\infty} \left\| \sum_{k=0}^{2^j-1} \sum_{\substack{Q\in\mathcal{D} \\ \pi_2(Q)=I_{j,k}}} (\chi_{I_{j,k}} \circ F) \cdot T[(\chi_{\pi_1(Q)} \circ F) \cdot f] \right\|_{L^q(\mathbb{R};Y)}^{q\cdot\frac{1}{q}} . \tag{10}$$

In the summations within the norm in the right-hand side of (9), note that any two distinct terms represent integrals with disjoint supports. As a result,

$$\begin{aligned} \|\tilde{T}f\|_{L^q(\mathbb{R};Y)} &\leq \sum_{j=1}^{\infty} \left[\sum_{k=0}^{2^j-1} \sum_{\substack{Q\in\mathcal{D} \\ \pi_2(Q)=I_{j,k}}} \|(\chi_{I_{j,k}} \circ F) \cdot T[(\chi_{\pi_1(Q)} \circ F) \cdot f]\|_{L^q(\mathbb{R};Y)}^q \right]^{\frac{1}{q}} \\ &\lesssim \sum_{j=1}^{\infty} \left[\sum_{k=0}^{2^j-1} \sum_{\substack{Q\in\mathcal{D} \\ \pi_2(Q)=I_{j,k}}} \|(\chi_{\pi_1(Q)} \circ F) \cdot f\|_{L^p(\mathbb{R};X)}^q \right]^{\frac{1}{q}} , \end{aligned}$$

where in the last step, we used the boundedness of T.

By the definition of F, if $l(Q) = 2^{-j}$, then

$$\|(\chi_{\pi_1(Q)} \circ F) \cdot f\|_{L^p(\mathbb{R};X)}^p = 2^{-j}.$$

Moreover, by part (4) of Lemma 4, for any $j \geq 0$ and $0 \leq k < 2^j$, there are at most five $Q \in \mathcal{D}$ such that $\pi_2(Q) = I_{j,k}$. Combining these observations, we have

$$\|\tilde{T}f\|_{L^q(\mathbb{R};Y)} \lesssim \sum_{j=1}^{\infty} \left(\sum_{k=0}^{2^j-1} 2^{-\frac{qj}{p}} \right)^{\frac{1}{q}} \leq \sum_{j=0}^{\infty} (2^{-\frac{qj}{p}} 2^j)^{\frac{1}{q}} \leq \sum_{j=0}^{\infty} 2^{j(\frac{1}{q}-\frac{1}{p})}.$$

Since $p < q$, then $q^{-1} - p^{-1} < 0$, and it follows that

$$\|\tilde{T}f\|_{L^q(\mathbb{R};Y)} < \infty,$$

as desired. This completes the proof of Theorem 2.

3. Further Extensions

Finally, we discuss some extensions and generalizations of Theorems 1 and 2.

3.1. **The Case $q = \infty$.** First, we can extend Theorem 2 to the case $q = \infty$.

Theorem 6. *Let X and Y denote Banach spaces, and let*

$$T : L^p(\mathbb{R}; X) \to L^\infty(\mathbb{R}; Y), \qquad 1 \leq p \leq \infty,$$

such that T can be expressed as an integral transform,

$$Tf(t) = \int_{\mathbb{R}} K(t,s) f(s) ds, \qquad K : \mathbb{R} \times \mathbb{R} \to B(X,Y).$$

If $\tilde{T}$ is defined as in the statement of Theorem 2, then $\tilde{T}$ extends to a bounded linear operator mapping from $L^p(\mathbb{R}; X)$ into $L^\infty(\mathbb{R}; Y)$.

Remark. *In contrast to the statement of Theorem 2, the exponent p is allowed to be the same as $q = \infty$ in the statement of Theorem 6.*

Proof. For any $t \in \mathbb{R}$, we can write $\tilde{T}f(t)$ as

$$|\tilde{T}f(t)| = \left| \int_{-\infty}^{\infty} K(t,s)[\chi_{(-\infty,t]}(s)f(s)]ds \right| = |T[\chi_{(-\infty,t]}f](t)|.$$

From our boundedness assumption for T, we now have

$$|\tilde{T}f(t)| \leq \|T[\chi_{(-\infty,t]}f]\|_{L^\infty(\mathbb{R};Y)} \lesssim \|\chi_{(-\infty,t]}f\|_{L^p(\mathbb{R};X)} \lesssim \|f\|_{L^p(\mathbb{R};X)}.$$

This completes the proof of Theorem 6. □

Furthermore, if, assuming the same setting as in Theorem 6, we define

$$\hat{T}g(t) = \int_t^\infty K(t,s)g(s)ds,$$

then an analogous argument shows $\hat{T}$ is also bounded from $L^p(\mathbb{R};X)$ into $L^\infty(\mathbb{R};Y)$.

3.2. **The Case** $p = 1$**.** Using the $q = \infty$ case in Theorem 6 along with a standard duality argument, we can prove an analogue for the opposite case, $p = 1$.

Theorem 7. *Let X and Y denote Banach spaces, and let*

$$T : L^1(\mathbb{R};X) \to L^q(\mathbb{R};Y), \qquad 1 \leq q \leq \infty,$$

such that T can be expressed as an integral transform,

$$Tf(t) = \int_{\mathbb{R}} K(t,s)f(s)ds, \qquad K : \mathbb{R} \times \mathbb{R} \to B(X,Y),$$

If $\tilde{T}$ is defined as in the statement of Theorem 2, then $\tilde{T}$ extends to a bounded linear operator mapping from $L^1(\mathbb{R};X)$ into $L^q(\mathbb{R};Y)$.

Proof. Let q' satisfy $q^{-1} + (q')^{-1} = 1$, so that the adjoint S of T is a bounded linear operator from $L^{q'}(\mathbb{R};Y)$ into $L^\infty(\mathbb{R};X)$. Moreover, for appropriate f and g,

$$\begin{aligned}
\int_{-\infty}^{\infty} Sg(t)f(t)dt &= \int_{-\infty}^{\infty} g(t)Tf(t)dt \\
&= \int_{-\infty}^{\infty} g(t) \left[\int_{-\infty}^{\infty} K(t,s)f(s)ds \right] dt \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} K(s,t)g(s)ds \right] f(t)dt.
\end{aligned}$$

As a result, S is of the same form as T, i.e.,

$$Sg(t) = \int_{-\infty}^{\infty} K(s,t)g(s)ds.$$

Next, we consider the adjoint U of $\tilde{T}$:

$$\begin{aligned}
\int_{-\infty}^{\infty} Ug(t)f(t)dt &= \int_{-\infty}^{\infty} g(t)\tilde{T}f(t)dt \\
&= \int_{-\infty}^{\infty} g(s) \left[\int_{-\infty}^{s} K(s,t)f(t)dt \right] ds \\
&= \int_{-\infty}^{\infty} \left[\int_{t}^{\infty} K(s,t)g(s)ds \right] f(t)dt.
\end{aligned}$$

It follows that U is precisely $\hat{S}$, as defined in the discussion after the proof of Theorem 6. Since S is bounded from $L^{q'}(\mathbb{R};Y)$ into $L^\infty(\mathbb{R};X)$, then so is U. Consequently, by duality, $\tilde{T}$ is also bounded from $L^1(\mathbb{R};X)$ into $L^q(\mathbb{R};X)$. □

3.3. **The General Estimate.** The original paper by Christ and Kiselev, [1], actually proved a more general estimate for maximal-type operators. In this generalized estimate, the operator T now acts on arbitrary measure spaces, i.e.,

$$T : L^p(X,\mu) \to L^q(Y,\nu).$$

In addition, we consider monotonic families of characteristic functions χ_α, parametrized by $\alpha \in \mathbb{R}$, satisfying certain conditions.

We now define $\tilde{T}$ to be the following maximal operator:

$$\tilde{T}g = \sup_{\alpha\in\mathbb{R}} T(\chi_\alpha f).$$

The main theorems now state that if T is bounded from L^p to L^q, like in our previous estimates, then $\tilde{T}$ is also bounded from L^p to L^q.

In particular, all our Theorems can be recovered from this general estimate by taking T as before, and by defining $\chi_\alpha = \chi_{(-\infty,\alpha]}$.

References

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