HÖRMANDER'S INEQUALITY FOR WAVE EQUATIONS

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1. Introduction

These notes contain a detailed proof of Hörmander's inequality for wave equations in $(1+3)$-dimensions, which can be found in [2]. [1] This estimate was an essential component in [1, 3], which established small data global existence for certain nonlinear wave equations in $(1+3)$-dimensions.

Let $\mathbb{R}^{1+3}$ denote Minkowski spacetime, and let ∂_0, ∂_1, ∂_2, ∂_3 denote the standard coordinate vector fields, where "0", as usual, denotes the time component. Furthermore, we define the following vector fields, which generate the conformal symmetries of the Minkowski spacetime $\mathbb{R}^{1+3}$:

- *Translation*: for any $0 \leq \alpha \leq 3$, the vector field ∂_α.
- *Rotations* and *boosts*: for any $0 \leq \alpha, \beta \leq 3$, the vector field

$$\Omega_{\alpha\beta} = c_\beta x^\beta \partial_\alpha - c_\alpha x^\alpha \partial_\beta, \qquad c_\mu = \begin{cases} -1 & \mu = 0, \\ 1 & \mu > 0. \end{cases}$$

- *Dilation*: the vector field

$$L_0 = t\partial_0 + \sum_{i=1}^{3} x^i \partial_i.$$

Furthermore, we define the following notational conventions:

- Let Γ denote any of the above vector fields.
- Let $\dot{\Gamma}$ denote any one of L_0 or the $\Omega_{\alpha\beta}$'s (the homogeneous vector fields).
- Let $\ddot{\Gamma}$ denote any one of Ω_{ij}'s, for $1 \leq i, j \leq 3$ (the spatial rotations).

We will also use multi-indices to denote compositions of the above vector fields.

The main inequality can now be stated as follows.

Theorem 1. *Let $F \in C^2((0,\infty) \times \mathbb{R}^3)$, and suppose $u : [0,\infty) \times \mathbb{R}^3 \to \mathbb{R}$ satisfies*

$$\Box u = F, \qquad u|_{t=0} \equiv 0, \qquad \partial_t u|_{t=0} \equiv 0.$$

Then, the following inequality holds for any $t > 0$ and $x \in \mathbb{R}^3$:

$$(1) \qquad (1 + t + |x|)|u(t,x)| \lesssim \sum_{|\alpha| \leq 2} \int_0^t \int_{\mathbb{R}^3} \frac{|\Gamma^\alpha F(s,y)|}{1 + s + |y|} dy ds.$$

[1] Thanks to Yannis Angelopoulos for the correct references.

1.1. **Preliminaries.** Recall that, with u and F as given by the hypotheses of Theorem 1, we have the following explicit equation for u in terms of F:

$$u(t,x) = \frac{1}{4\pi} \int_{|y|<t} \frac{F(t-|y|, x-y)}{|y|} dy, \qquad t > 0 \quad x \in \mathbb{R}^3. \tag{2}$$

The integral on the right-hand side is over a disk in $\mathbb{R}^3$. Recall in addition the *strong Huygens principle*, which implies that if F vanishes on the past null cone segment starting at (t,x) and ending at $t=0$, then $u(t,x)$ is also zero.

If F is spherically symmetric, i.e., $F(s,y) = F^*(s,|y|)$, then (2) implies that u is also spherically symmetric, $u(t,x) = u^*(t,|x|)$. Furthermore, from a direct computation using (2), one can derive the formula

$$ru^*(t,r) = \frac{1}{2} \int_0^t \int_{|r-(t-s)|}^{r+(t-s)} F^*(s,\rho) \rho d\rho ds. \tag{3}$$

For details behind this calculation, see [4].

We also require the following algebraic observation: in the region $2|x| \leq t$ (which is in particular away from the null cone $|x| = t$), we have the bound

$$t|\partial_\mu f(t,x)| \lesssim \sum_{|\alpha|=1} |\dot{\Gamma}^\alpha f(t,x)|, \qquad 0 \leq \mu \leq 3, \quad 2|x| \leq t.$$

The proof relies on explicit representations of $(t^2 - |x|^2)\partial_\mu$ as linear combinations of the $\dot{\Gamma}$'s, and by the observation that $t - |x| \simeq t$ in the region $2|x| \leq t$. By an induction argument and by the observation that the coefficients of the $\dot{\Gamma}$'s are homogeneous, we obtain for any multi-index β the more general estimate

$$t^{|\beta|} |\partial^\beta f(t,x)| \lesssim \sum_{1 \leq |\alpha| \leq |\beta|} |\dot{\Gamma}^\alpha f(t,x)|, \qquad 2|x| \leq t. \tag{4}$$

Again, the reader is referred to [4] for details.

Finally, we will need the following estimate, for which the proof can be found in [4]: if $\varphi \in C^1(\mathbb{R})$ has compact support, then

$$\int_0^\infty |\varphi(r)| r dr \lesssim \int_0^\infty |\varphi'(r)| r^2 dr. \tag{5}$$

Now, if $f \in C^1(\mathbb{R}^3)$ has compact support, then using polar coordinates,

$$\int_{\mathbb{R}^3} \frac{|f(x)|}{|x|} dx = \int_{\mathbb{S}^2} \int_0^\infty |f(r\omega)| r dr d\omega = \int_{\mathbb{S}^2} \int_0^\infty |\partial_r f(r\omega)| \, r^2 dr d\omega.$$

Switching back to Cartesian coordinates, we have obtained

$$\int_{\mathbb{R}^3} \frac{|f(x)|}{|x|} dx \lesssim \int_{\mathbb{R}^3} |\nabla_x f(x)| dx. \tag{6}$$

2. From the Homogeneous Estimate

The first step in proving Theorem 1 is to reduce (1) to the following *homogeneous* estimate, which holds whenever F is supported away from the origin.

Lemma 2. *Assume the hypotheses of Theorem 1, and suppose in addition that*

$$\operatorname{supp} F \subseteq \{(s,y) \mid s + |y| \geq C\},$$

for some $C > 0$. Then, for any $t > 0$ and $x \in \mathbb{R}$,

$$(1 + |x|)|u(1, x)| \lesssim \sum_{|\alpha| \leq 2} \int_0^1 \int_{\mathbb{R}^3} \frac{|\dot{\Gamma}^\alpha F(s, y)|}{s + |y|} dyds. \tag{7}$$

We defer the proof of Lemma 2 until the next section. In this section, we assume Lemma 2, and we show how Theorem 1 can be obtained from this.

2.1. **Scaling Symmetry.** Because of the scaling symmetry associated with the linear wave equation, we can immediately generalize Lemma 2 to the following.

Lemma 3. *Assume the hypotheses of Theorem 1, and suppose in addition that*

$$\operatorname{supp} F \subseteq \{(s, y) \mid s + |y| \geq 1\}.$$

Then, for any $t > 0$ and $x \in \mathbb{R}$,

$$(t + |x|)|u(t, x)| \lesssim \sum_{|\alpha| \leq 2} \int_0^t \int_{\mathbb{R}^3} \frac{|\dot{\Gamma}^\alpha F(s, y)|}{s + |y|} dyds. \tag{8}$$

Proof. Define the functions $\bar{u}$ and $\bar{F}$ by

$$\bar{u}(\bar{t}, \bar{x}) = u(t\bar{t}, t\bar{x}), \qquad \bar{F}(\bar{s}, \bar{y}) = t^2 F(t\bar{s}, t\bar{y}).$$

Note that $\bar{F}$ is supported within the region $\bar{s} + |\bar{y}| \geq t^{-1}$. By scaling symmetry, $\Box \bar{u} = \bar{F}$, with vanishing initial data, and hence by (7),

$$\begin{aligned} (1 + t^{-1}|x|)|u(t, x)| &= (1 + t^{-1}|x|)|\bar{u}(1, t^{-1}x)| \\ &\lesssim \sum_{|\alpha| \leq 2} \int_0^1 \int_{\mathbb{R}^3} \frac{|\dot{\Gamma}^\alpha \bar{F}(\bar{s}, \bar{y})|}{\bar{s} + |\bar{y}|} d\bar{y}d\bar{s} \end{aligned}$$

Since the $\dot{\Gamma}$'s are homogeneous, the numerator of the integrand satisfies

$$\dot{\Gamma}^\alpha \bar{F}(\bar{s}, \bar{y}) = \dot{\Gamma}^\alpha|_{(\bar{s}, \bar{y})}[t^2 F(t\bar{s}, t\bar{y})] = t^2 \dot{\Gamma}^\alpha F(t\bar{s}, t\bar{y}).$$

Combining the above with the change of variables $s = t\bar{s}$, $y = t\bar{y}$, we have

$$\begin{aligned} (1 + t^{-1}|x|)|u(t, x)| &\lesssim t^2 \sum_{|\alpha| \leq 2} \int_0^1 \int_{\mathbb{R}^3} \frac{|\dot{\Gamma}^\alpha F(t\bar{s}, t\bar{y})|}{\bar{s} + |\bar{y}|} d\bar{y}d\bar{s} \\ &\lesssim t^{-1} \sum_{|\alpha| \leq 2} \int_0^t \int_{\mathbb{R}^3} \frac{|\dot{\Gamma}^\alpha F(s, y)|}{s + |y|} dyds. \end{aligned}$$

Multiplying both sides by t completes the proof. □

2.2. **Translation Symmetry.** Next, using Lemma 3 along with the translation symmetry associated with the linear wave equation, we can handle the remaining case, in which F is supported near the origin.

Lemma 4. *Assume the hypotheses of Theorem 1, and suppose in addition that*

$$\operatorname{supp} F \subseteq \{(s, y) \mid s + |y| \leq 2\}.$$

Then, the following inequality holds:

$$(1 + t + |x|)|u(t, x)| \lesssim \sum_{|\alpha| \leq 2} \int_0^t \int_{\mathbb{R}^3} \frac{|\Gamma^\alpha F(s, y)|}{1 + s + |y|} dyds. \tag{9}$$

Proof. Consider the function u', defined

$$u'(t,x) = u(t, x + 8e_1), \qquad e_1 = (1,0,0).$$

Then, u' satisfies the wave equation

$$\Box u' = F', \qquad F'(t,x) = F(t, x + 8e_1).$$

In particular, F' is now supported in the region $s + |y| \geq 1$, so by (8),

$$\begin{aligned} (t + |x|)|u'(t,x)| &\lesssim \sum_{|\alpha| \leq 2} \int_0^t \int_{\mathbb{R}^3} \frac{\dot{\Gamma}^\alpha F'(s,y)}{s + |y|} dy ds \\ &= \sum_{|\alpha| \leq 2} \int_0^t \int_{\mathbb{R}^3} \frac{\dot{\Gamma}^\alpha|_{(s,y)}[F(s, y + 8e_1)]}{s + |y|} dy ds. \end{aligned}$$

Now, the operators $\dot{\Gamma}$ on the right-hand side are applied at the point (s,y), while F is applied at the point $(s, y + 8e_1)$. To apply $\dot{\Gamma}$ at $(s, y + 8e_1)$ instead, one picks up extra terms of the form $\partial_1 F(s, y + 8e_1)$. As a result,

$$\begin{aligned} (t + |x|)|u'(t,x)| &\lesssim \sum_{|\alpha| \leq 2} \int_0^t \int_{\mathbb{R}^3} \frac{\Gamma^\alpha F(s, y + 8e_1)}{s + |y|} dy ds \\ &\lesssim \sum_{|\alpha| \leq 2} \int_0^t \int_{\mathbb{R}^3} \frac{\Gamma^\alpha F(s,y)}{s + |y - 8e_1|} dy ds. \end{aligned}$$

Since F is supported on $s + |y| \leq 1$, then $|y - 8e_1| \gtrsim 1$, and hence

$$s + |y - 8e_1| \simeq 1 \simeq 1 + s + |y|.$$

Consequently, we have

$$(t + |x|)|u'(t,x)| \lesssim \sum_{|\alpha| \leq 2} \int_0^t \int_{\mathbb{R}^3} \frac{\Gamma^\alpha F(s,y)}{1 + s + |y|} dy ds.$$

Since the above is true for all $x \in \mathbb{R}^3$, we can change variables and obtain

$$(t + |x - 8e_1|)|u(t,x)| \lesssim \sum_{|\alpha| \leq 2} \int_0^t \int_{\mathbb{R}^3} \frac{\Gamma^\alpha F(s,y)}{1 + s + |y|} dy ds.$$

If $|x - 8e_1| \leq 1$, then $7 \leq |x| \leq 9$, and by the strong Huygens principle, since F is supported on $s + |y| \leq 1$, then $u(t,x)$ is nonzero only when $t \simeq |x|$. Thus,

$$t + |x - 8e_1| \gtrsim t \simeq 1 \simeq 1 + t + |x|$$

in this case. On the other hand, if $|x - 8e_1| \geq 1$ and $|x| \leq 16$, then

$$t + |x - 8e_1| \gtrsim 1 + t \simeq 1 + t + |x|.$$

Finally, if $|x - 8e_1| \geq 1$ and $|x| \geq 16$, then $|x - 8e_1| \simeq |x| \simeq 1 + |x|$, and hence

$$t + |x - 8e_1| \simeq 1 + t + |x|.$$

This covers all possible cases, the combination of which yields

$$\begin{aligned} (1 + t + |x|)|u(t,x)| &\lesssim (t + |x - 8e_1|)|u(t,x)| \\ &\lesssim \sum_{|\alpha| \leq 2} \int_0^t \int_{\mathbb{R}^3} \frac{\Gamma^\alpha F(s,y)}{1 + s + |y|} dy ds. \end{aligned}$$

□

2.3. **Completion of the Proof.** Combining Lemmas 3 and 4, we can complete the proof of Theorem 1. Consider general F, as in Theorem 1. Using a cutoff function, we can split F as $F = F_h + F_l$, where F_h and F_l are supported on the regions $s + |y| \geq 1$ and $s + |y| \leq 2$, respectively. Next, write $u = u_h + u_l$, where $\Box u_h = F_h$, $\Box u_l = F_l$, and both u_h and u_l have zero data at $t = 0$.

By applying Lemma 4, we can write

$$\begin{aligned}(1 + t + |x|)|u(t, x)| &\leq (1 + t + |x|)|u_h(t, x)| + (1 + t + |x|)|u_l(t, x)| \\ &\lesssim (1 + t + |x|)|u_h(t, x)| + \sum_{|\alpha|\leq 2} \int_0^t \int_{\mathbb{R}^3} \frac{|\Gamma^\alpha F_l(s, y)|}{1 + s + |y|} dyds.\end{aligned}$$

Furthermore, from the strong Huygens principle, we can see that $u_h(t, x)$ is nonzero only when $t + |x| \gtrsim 1$. Combining this with Lemma 3 and the fact that F_h is supported in the region $s + |y| \geq 1$, we have

$$\begin{aligned}(1 + t + |x|)|u_h(t, x)| &\lesssim (t + |x|)|u_h(t, x)| \\ &\lesssim \sum_{|\alpha|\leq 2} \int_0^t \int_{\mathbb{R}^3} \frac{|\dot{\Gamma}^\alpha F_h(s, y)|}{s + |y|} dyds \\ &\lesssim \sum_{|\alpha|\leq 2} \int_0^t \int_{\mathbb{R}^3} \frac{|\dot{\Gamma}^\alpha F_h(s, y)|}{1 + s + |y|} dyds.\end{aligned}$$

Combining the above, we obtain

$$(1 + t + |x|)|u(t, x)| \lesssim \sum_{|\alpha|\leq 2} \int_0^t \int_{\mathbb{R}^3} \frac{|\Gamma^\alpha F_h(s, y)| + |\Gamma^\alpha F_l(s, y)|}{1 + s + |y|} dyds.$$

Finally, since $F_h = \varphi_h F$ and $F_l = \varphi_l F$ for some cutoff functions φ_h and φ_l, and since any derivative of φ_h and φ_l is supported entirely on $s + |y| \simeq 1$, then

$$(1 + t + |x|)|u(t, x)| \lesssim \sum_{|\alpha|\leq 2} \int_0^t \int_{\mathbb{R}^3} \frac{|\Gamma^\alpha F(s, y)|}{1 + s + |y|} dyds,$$

as desired. This completes the proof of Theorem 1.

3. The Homogeneous Estimate

It remains to prove the homogeneous estimate of Lemma 2, which is the objective of this section. To do this, we once again break into cases.

Lemma 5. *Assume the hypotheses of Theorem 1, and suppose in addition that*

$$\operatorname{supp} F \subseteq \{(s, y) \mid s + |y| \geq C\}, \qquad C > 0.$$

- *If* $\operatorname{supp} F$ *is contained also in the region* $2|y| \leq s$, *then*

$$(1 + |x|)|u(1, x)| \lesssim \sum_{|\alpha|\leq 2} \int_0^1 \int_{\mathbb{R}^3} \frac{|\dot{\Gamma}^\alpha F(s, y)|}{s} dyds. \tag{10}$$

- *If* $\operatorname{supp} F$ *is contained also in the region* $3|y| \geq s$, *then*

$$(1 + |x|)|u(1, x)| \lesssim \sum_{|\alpha|\leq 2} \int_0^1 \int_{\mathbb{R}^3} \frac{|\dot{\Gamma}^\alpha F(s, y)|}{|y|} dyds. \tag{11}$$

3.1. **Proof of Lemma 2.** Let us first assume Lemma 5; we use this now to prove Lemma 2. Let F be as in the hypotheses of Lemma 2, and decompose

$$F(s,y) = F_a(s,y) + F_b(s,y) = \psi(s^{-2}|y|^2)F(s,y) + [1 - \psi(s^{-2}|y|^2)]F(s,y),$$

where ψ is a cutoff function defined on $\mathbb{R}$, where F_a is supported in the region $2|y| \leq s$, and F_b is supported in the region $3|y| \geq s$.

Since any derivative of $\Psi(s,y) = \psi(s^{-2}|y|^2)$ is supported in the region $s \simeq |y|$,

$$|\dot{\Gamma}\Psi(s,y)| \lesssim (s + |y|)|\partial\Psi(s,y)| \lesssim (s + |y|)\frac{1}{s+|y|}\|\psi'\|_{L^\infty} \lesssim 1.$$

where $\dot{\Gamma}$ is any one of the homogeneous vector fields. Furthermore, by induction,

$$|\dot{\Gamma}^\alpha\Psi(s,y)| \lesssim 1,$$

for any multi-index α, where the constant depends on ψ itself. As a result,

$$\sum_{|\alpha|\leq 2}[|\dot{\Gamma}^\alpha F_a(s,y)| + |\dot{\Gamma}^\alpha F_b(s,y)|] \lesssim \sum_{|\alpha|\leq 2}|\dot{\Gamma}^\alpha F(s,y)|. \tag{12}$$

Next, we decompose $u = u_a + u_b$, where $\Box u_a = F_a$, where $\Box u_b = F_b$, and where both u_a and u_b have zero initial data at $t = 0$. By (10) and (11),

$$\begin{aligned}(1 + |x|)|u(1,x)| &\leq (1 + |x|)|u_a(1,x)| + (1 + |x|)|u_b(1,x)| \\ &\lesssim \sum_{|\alpha|\leq 2}\int_0^1\int_{\mathbb{R}^3}\frac{|\dot{\Gamma}^\alpha F_a(s,y)|}{s}dyds + \sum_{|\alpha|\leq 2}\int_0^1\int_{\mathbb{R}^3}\frac{|\dot{\Gamma}^\alpha F_b(s,y)|}{|y|}dyds.\end{aligned}$$

By our assumptions, $s \simeq s + |y|$ on the support of F_a, and $|y| \simeq s + |y|$ on the support of F_b. As a result, the above inequality becomes

$$\begin{aligned}(1 + |x|)|u(1,x)| &\lesssim \sum_{|\alpha|\leq 2}\int_0^1\int_{\mathbb{R}^3}\frac{|\dot{\Gamma}^\alpha F_a(s,y)| + |\dot{\Gamma}^\alpha F_b(s,y)|}{s + |y|}dyds \\ &\lesssim \sum_{|\alpha|\leq 2}\int_0^1\int_{\mathbb{R}^3}\frac{|\dot{\Gamma}^\alpha F(s,y)|}{s + |y|}dyds,\end{aligned}$$

where in the last step, we applied (12). This completes the proof of Lemma 2.

3.2. **Proof of** (10)**.** It remains to prove the two estimates (10) and (11) that comprise Lemma 5. We treat the first estimate (10) here.

Since F is supported in $2|y| \leq s$, then by the strong Huygens principle, we need only consider when $|x| \leq 1$. Thus, it suffices to show

$$|u(1,x)| \lesssim \sum_{|\alpha|\leq 2}\int_0^1\int_{\mathbb{R}^3}\frac{|\dot{\Gamma}^\alpha F(s,y)|}{s}dyds, \qquad |x| \leq 1. \tag{13}$$

If $1/2 \leq |y| < 1$, then by using the fundamental theorem of calculus and applying an appropriate cutoff function, we can estimate

$$F(1 - |y|, x - y) \lesssim \int_0^1 [|\partial_s F(s, x-y)| + |F(s, x-y)|]ds.$$

On the other hand, if $|y| < 1/2$, then $1 - |y| \geq 1/2$, and we can apply a similar estimate as before, but which avoids the region $s \ll 1$:

$$F(1 - |y|, x - y) \lesssim \int_{\frac{1}{2}}^1 [|\partial_s F(s, x-y)| + |F(s, x-y)|]ds.$$

Applying the representation formula (2) and the above, we have

$$|u(1,x)| \lesssim \int_{\frac{1}{2}}^{1} \int_{|y|<\frac{1}{2}} \frac{[|\partial_s F(s,x-y)| + |F(s,x-y)|]}{|y|} dyds + \int_{0}^{1} \int_{\frac{1}{2}\leq|y|<1} \frac{[|\partial_s F(s,x-y)| + |F(s,x-y)|]}{|y|} dyds$$

For the second term on the right-hand side, we note that $|y|^{-1} \simeq 1$, while for the first term on the right-hand side, we apply (6) and note that $s \simeq 1$. This yields

$$\begin{aligned} |u(1,x)| &\lesssim \int_{\frac{1}{2}}^{1} \int_{|y|<\frac{1}{2}} [s|\nabla_y \partial_s F(s,x-y)| + |\nabla_y F(s,x-y)|] dyds \\ &\quad + \int_{0}^{1} \int_{\frac{1}{2}\leq|y|<1} [|\partial_s F(s,x-y)| + |F(s,x-y)|] dyds \\ &\lesssim \int_{0}^{1} \int_{\mathbb{R}^3} [s|\nabla_y \partial_s F(s,y)| + |\partial_s F(s,y)| + |\nabla_y F(s,y)| + |F(s,y)|] dyds \\ &\lesssim \int_{0}^{1} \int_{\mathbb{R}^3} \frac{s^2|\nabla_y \partial_s F(s,y)| + s|\partial_s F(s,y)| + s|\nabla_y F(s,y)| + |F(s,y)|}{s} dyds. \end{aligned}$$

Since F is supported away from the null cone, then by (4),

$$|u(1,x)| \leq \int_{0}^{1} \int_{\mathbb{R}^3} \frac{\sum_{|\alpha|\leq 2} |\dot{\Gamma}^\alpha F(s,y)|}{s} dyds,$$

which proves (13), and hence (10).

3.3. **Proof of** (11)**, if** $|x| \geq 1/4$**.** In this case, it suffices to show that

$$|x||u(1,x)| \lesssim \sum_{|\alpha|\leq 2} \int_{0}^{1} \int_{\mathbb{R}^3} \frac{|\dot{\Gamma}^\alpha F(s,y)|}{|y|} dyds. \tag{14}$$

Let $M : (0,\infty) \times [0,\infty)$ be given by

$$M(s,\rho) = \sup_{\omega\in\mathbb{S}^2} |F(s,\rho\omega)|.$$

Applying the Sobolev estimate on $\mathbb{S}^2$ yields the bound

$$M(s,\rho) \lesssim \sum_{|\alpha|\leq 2} \int_{\mathbb{S}^2} |\ddot{\Gamma}^\alpha F(s,\rho\omega)| d\omega,$$

since the spatial rotation vector fields Ω_{ij} generate all the directional derivatives on $\mathbb{S}^2$. Integrating the above over s and ρ, we obtain

$$\begin{aligned} \int_{0}^{1} \int_{0}^{\infty} M(s,\rho)\rho d\rho ds &\lesssim \int_{0}^{1} \int_{0}^{\infty} \int_{\mathbb{S}^2} \sum_{|\alpha|\leq 2} |\ddot{\Gamma}^\alpha F(s,\rho\omega)| d\omega\rho d\rho ds \\ &\lesssim \sum_{|\alpha|\leq 2} \int_{0}^{1} \int_{\mathbb{R}^3} \frac{|\ddot{\Gamma}^\alpha F(s,y)|}{|y|} dyds. \end{aligned}$$

Next, suppose U is the solution of $\Box U(t,x) = M(t,|x|)$, with zero initial data. Comparing the representation formula (2) for both u and U, we see that

$$|x||u(1,x)| \leq |x|U(1,x), \qquad x \in \mathbb{R}^3.$$

Moreover, U is spherically symmetric, and applying (3) to U yields

$$\begin{aligned}|x||u(1,x)| &\lesssim \int_0^1 \int_{|r-(1-s)|}^{r+(1-s)} M(s,\rho)\rho d\rho ds \\ &\lesssim \int_0^1 \int_0^\infty M(s,\rho)\rho d\rho ds \\ &\lesssim \sum_{|\alpha|\leq 2} \int_0^1 \int_{\mathbb{R}^3} \frac{|\ddot{\Gamma}^\alpha F(s,y)|}{|y|} dyds.\end{aligned}$$

This completes the proof of (14), and hence (11), whenever $|x| \geq 1/4$.

3.4. **Proof of** (11)**, if** $|x| \leq 1/4$**.** In this case, we need only show

$$|u(1,x)| \lesssim \int_0^1 \int_{|y|<2} [|L_0 F(s,y)| + |F(s,y)|] dyds, \tag{15}$$

since $|x| \lesssim 1$, and since $1 \lesssim |y|^{-1}$ on the domain $|y| < 2$.

By our assumptions on $\operatorname{supp} F$, we see that if $(1-|w|, x-w) \in \operatorname{supp} F$, then

$$3|x-w| > 1-|w|, \qquad 4|w| > 1-3|x| > \frac{1}{4}.$$

As a result, (2) yields

$$|u(1,x)| \lesssim \int_{\frac{1}{16}<|w|<1} |F(1-|w|, x-w)| dw.$$

Moreover, using a cutoff function and the fundamental theorem of calculus,

$$\begin{aligned}|u(1,x)| &\lesssim \int_1^{\frac{16}{15}} \int_{\frac{1}{16}<|w|<1} |\partial_\tau [F(\tau(1-|w|), \tau(x-w))]| dwd\tau \\ &\qquad + \int_1^{\frac{16}{15}} \int_{\frac{1}{16}<|w|<1} |F(\tau(1-|w|), \tau(x-w))| dwd\tau \\ &= \int_1^{\frac{16}{15}} \int_{\frac{1}{16}<|w|<1} \tau^{-1} |L_0 F(\tau(1-|w|), \tau(x-w))| dwd\tau \\ &\qquad + \int_1^{\frac{16}{15}} \int_{\frac{1}{16}<|w|<1} |F(\tau(1-|w|), \tau(x-w))| dwd\tau \\ &\lesssim \int_1^{\frac{16}{15}} \int_{\frac{1}{16}<|w|<1} |L_0 F(\tau(1-|w|), \tau(x-w))| dwd\tau \\ &\qquad + \int_1^{\frac{16}{15}} \int_{\frac{1}{16}<|w|<1} |F(\tau(1-|w|), \tau(x-w))| dwd\tau.\end{aligned}$$

We now adopt the change of variables

$$s(\tau, w) = \tau(1-|w|), \qquad y(\tau, w) = \tau(x-w).$$

The Jacobian of this transformation is

$$\left| \frac{\partial(s,\tau)}{\partial(\tau,w)} \right| = \left| \det \begin{bmatrix} 1-|w| & -\frac{w^1}{|w|} & -\frac{w^2}{|w|} & -\frac{w^3}{|w|} \\ x^1-w^1 & -1 & 0 & 0 \\ x^2-w^2 & 0 & -1 & 0 \\ x^3-w^3 & 0 & 0 & -1 \end{bmatrix} \right|$$

$$= \left| |w| - 1 + \frac{w^1 w^1}{|w|} - \frac{x^1 w^1}{|w|} + \frac{w^2 w^2}{|w|} - \frac{x^2 w^2}{|w|} + \frac{w^3 w^3}{|w|} - \frac{x^3 w^3}{|w|} \right|$$
$$= \left| -1 + \frac{x \cdot w}{w} \right|.$$

Since $|x| \leq 1/4$ by assumption, then

$$\left| \frac{\partial(s, \tau)}{\partial(\tau, w)} \right| \simeq 1$$

for all τ and w in our domain of consideration.

Applying this change of variables to the τ-w-integrals and recalling the above comparison for the associated Jacobian, then we obtain

$$|u(1, x)| \lesssim \int_0^1 \int_{|y|<2} [|L_0 F(s, y)| + |F(s, y)|] dy ds,$$

which is precisely our desired bound (15).

References

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