

# A BASIC GUIDE TO COMPLEX VARIABLES

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**Complex analysis** is the area of mathematics dealing with calculus on the complex plane. Unfortunately, the subject can be a tough sell to first-time students, thanks to a lack of motivation for working with a seemingly unnatural number system. However, complex analysis has grown over the years into a powerful subject. Even at its most basic level, one can find some of the most beautiful results in all of mathematics. Moreover, there are so many different ways complex numbers can be applied to other areas. For those who overcome the initial learning curve, complex analysis can be an enriching and rewarding experience.

These notes were created by piecing together the various lectures I gave for the course **MAT334: Complex Variables** at the **University of Toronto**, both in **fall 2012** and in **spring 2013**. In these notes, we will cover many of the topics present within an elementary undergraduate complex analysis course. We assume the reader is familiar with the real numbers, in particular with many of the basic notions in precalculus and calculus. On the other hand, we will not assume any familiarity with the imaginary and complex number systems.

**Remark.** Unfortunately, these notes current lack exercise problems, as the current focus is on explaining basic ideas and providing examples. For practice problems, consult your nearest real textbook.

## 1. THE COMPLEX NUMBERS

This section deals with the basic properties of the complex plane. The first goal is motivational: to discuss the need to look beyond the real number line. From these guiding principles, we construct the complex numbers as an extension of the real number system. In the remainder of the chapter, we explore various basic properties of our newly created complex number system.

Throughout, we will use the following common notations for number systems:

**Definition 1.1.** We denote the set of **integers** by  $\mathbb{Z}$ , i.e.,

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

Moreover, we denote the set of **real numbers** by  $\mathbb{R}$ .

Furthermore, we will use the following common notations for describing sets:

**Definition 1.2.** We use the symbol “ $\in$ ” to denote membership in a set or collection—for example,  $x \in \mathbb{R}$  means that “ $x$  is a real number”.

We often describe sets as follows: we write  $\{x \in A \mid P\}$  to mean the set of all elements  $x$  of  $A$  which satisfies the condition  $P$ . For example,  $\{x \in \mathbb{Z} \mid x > 0\}$  is the set of all positive integers, i.e., the **natural numbers**.

**1.1. The Real Number System.** Before jumping into the complex numbers, let us begin our discussion first with a familiar topic, the real number system. The goal is to find motivation for constructing the complex numbers that we will study.

Rather than delving into the theoretical question of how  $\mathbb{R}$  is *constructed* (this is an interesting discussion in its own right), we ask instead the following question:

*What we can do with the real numbers?*

The first thing we can do with real numbers is to order them. Given two numbers  $x, y \in \mathbb{R}$ , we can compare them and determine exactly one of the following is true:

- (1)  $x$  is equal to  $y$ , i.e.,  $x = y$ .
- (2)  $x$  is less than  $y$ , i.e.,  $x < y$ .
- (3)  $x$  is greater than  $y$ , i.e.,  $x > y$ .

You may remember the above from high-school algebra as the **trichotomy property** (though if you don't, hopefully you still do remember the symbols  $=$ ,  $<$ , and  $>$ ). In addition, we have following **transitive property**:

- If real numbers  $x, y, z$  satisfy  $x \leq y$  and  $y \leq z$ , then  $x \leq z$  as well.

This trichotomy and transitive properties leads to the visual interpretation of  $\mathbb{R}$  as a line, i.e., the *real line*. Different real numbers are represented as different points on this line; moreover, if  $x < y$ , then  $x$  will lie to the left of  $y$  on this line.

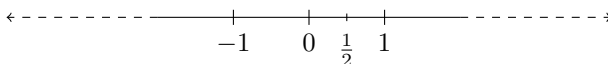


FIGURE 1. The real number line ... it's a line.

We can also perform algebraic operations on the real numbers. These include the following, which you should remember from basic algebra:

- **Addition:** Given real numbers  $x$  and  $y$ , we can add them:  $x + y$ .
- **Multiplication:** Given real numbers  $x$  and  $y$ , we can multiply them:  $x \cdot y$ .
- Given a real number  $x$ , we can find its **additive inverse**,  $-x$ .
- Moreover, if  $x \neq 0$ , we can find its **multiplicative inverse**,  $\frac{1}{x}$ .

Combining these operations allows us to subtract and divide real numbers:

$$x - y = x + (-y), \quad \frac{x}{y} = x \cdot \frac{1}{y}.$$

We can also *exponentiate* real numbers, though we have to be more careful:

- If  $k$  is a nonnegative integer, then we can take the  $k$ -th power of  $x$ :

$$x^k = x \cdots x \quad (k \text{ times}).$$

- If  $x \neq 0$ , we can also take negative integer powers of  $x$ :

$$x^{-k} = \left(\frac{1}{x}\right)^k.$$

- If  $x > 0$ , we can actually take any rational power of  $x$ :

$$x^{\frac{p}{q}} = \sqrt[q]{x^p}.$$

Most generally, for positive  $x$ , we can take any *real* power:

$$x^y = e^{y \cdot \ln x}, \quad y \in \mathbb{R}.$$



Again, all these things should be clear from high school algebra. *If not, then stop reading right this moment, review a few things, and return to this spot once you are finished!*

What is more interesting for our discussions here, though, is *what you cannot do with the real numbers*. We mentioned above that we can take any real power of a *positive* real number. However, this no longer holds if the base is negative. For example, if  $x < 0$ , then  $x^{1/2}$ , i.e., the square root of  $x$ , is not defined as a real number. This is also true for fourth roots, or for more exotic powers of  $x$ .

More generally, consider the problem of factoring a given real polynomial

$$p_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_0, \dots, a_n \in \mathbb{R},$$

into a product of monomials, i.e.,

$$p_n(x) = c(x - x_1)(x - x_2) \cdots (x - x_n), \quad x_1, \dots, x_n \in \mathbb{R}.$$

In other words, we want to find the *roots* of  $p_n$ , i.e., we want to find  $x_i$  such that  $p_n(x_i) = 0$ . It is clear that such factorizations *sometimes* exist; for a simple example when  $n = 2$ , you can consider the following algebraic identity:

$$x^2 - 1 = (x - 1)(x + 1).$$

On the other hand, if we consider instead  $x^2 + 1$ , then no such factorization exists. Indeed, factoring the above amounts to solving the equation  $x^2 = -1$ , that is, finding the square roots of  $-1$ . From the preceding discussion, we know that square roots of negative numbers do not exist, so at this point, you are surely doomed.

That the problems of finding square roots of negative numbers and of factoring polynomials fail to have solutions in  $\mathbb{R}$  is a fundamental shortcoming of the real line. There is no fancy and clever way to define a real number that would be the square root of  $-1$ , nor can you somehow magically completely factor  $x^2 + 1$ . The solutions to these problems inevitably do not exist. At this point, the question now becomes: “*So what do you do about this defect in the real number system?*”



**Option 1: Give up and go home.** If you want to think inside the box, then the answer to this question is simply to admit that since these are inherent defects of the real line, there really is nothing we can do. *Might as well pack it up now! We’re done here! Nothing more to see!* While this is certainly a viable option, this is also extremely unimaginative.



**Option 2: Bah! Let's make our own damn solution!**

If you wish to be more clever and think outside the box, then “fundamental defects” do not faze you. *What?  $-1$  has no square root? That's unacceptable! In that case, we'll make our own square root of  $-1$ ! While we're at it, we'll also define any real exponent of any negative number!*

By laughing in the face of adversity and choosing **Option 2**, we will systematically construct the complex numbers as we know it (or as you will know it). In other words, rather than moping because of a problem that cannot be solved, we instead “invent” a solution to this “impossible” problem, and we deal with the consequences of our actions later. While this may not seem like the most responsible way to deal with the real world, we will see in these notes that for our situation, this approach will prove to be not only useful but also quite profound.

Although we have summarized *what* we wanted to do, we have not yet addressed the question of *why*. The potential complaints are endless:

*“Why? I mean, these numbers aren't even real!”* (pun not intended)

*“Do they even mean anything in the real world?”*

*“Wahh!! I like to complain!”*

While the “meaning” of complex numbers may too ambiguous a notion for detailed discussion, what is indisputable is that complex analysis is extremely useful in many areas, ranging from something as simple as two-dimensional Euclidean geometry to far more advanced topics within, say, theoretical physics. Moreover, although many of these areas are not inherently tied to complex numbers (physics and engineering are concerned with the real world, after all), in practice, complex numbers often show up in essential, and sometimes surprising, ways in these disciplines.

While specific applications of complex analysis will have to wait until more background knowledge is gained, we can, for now, give a vague defense for why we adopt our current view. By looking “outside of” or “beyond” the real line, we essentially broaden our perspective by (literally) looking in another dimension beyond the single-dimensional real line. One hope, of course, is that this widened perspective will allow us to see and do more. What may be even more important, though, is that by taking a broader view, we can sometimes achieve a greater understanding of our original target, which in our case is the real line. We will give some examples of this in this chapter and throughout these notes.

**1.2. Complex Numbers.** At first glance, generating all exponents of negative real numbers seems like a tall task. There are millions, billions, *gazillions* of (or more accurately, infinitely many) new numbers to invent. To avoid being overwhelmed, we first focus our attention on the most basic subproblem: taking a square root of  $-1$ . Since this cannot be a real number, we *create a new object*, an *imaginary number* (because we just made it up) that we will *define* to be the square root of  $-1$ . In other words, we define a new non-real number  $i$  (or  $j$ , if you are an engineer, but let's not go there), which satisfies the algebraic property

$$i \cdot i = -1.$$

*Tada! We did it!* We now have our square root of  $-1$ ! However, there are still so many more numbers to be created. Moreover, this new number  $i$  is just sitting somewhere off the real line, being completely lonely and useless.

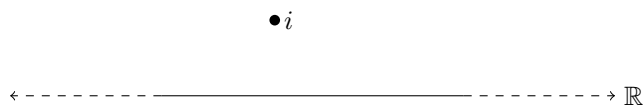


FIGURE 2. Well, that was pointless.

The point is that we do not just want to make up new numbers. We want to also be able to *do things* with these new numbers. Most importantly, we want the same *algebraic properties* for our new numbers as for the real numbers, i.e., we want to be able to add, multiply, and take inverses of our new numbers in the same manner that we did for the real numbers. In particular, *we want our operations to have the same laws as the corresponding operations for the real numbers*. To review a bit of high school algebra, the main laws for the real numbers are as follows:

- **Commutativity:** If  $x, y \in \mathbb{R}$ , then

$$x + y = y + x, \quad xy = yx.$$

- **Associativity:** If  $x, y, p \in \mathbb{R}$ , then

$$x + (y + p) = (x + y) + p, \quad x \cdot (y \cdot p) = (x \cdot y) \cdot p.$$

- **Identity:** If  $x \in \mathbb{R}$ , then

$$x + 0 = x, \quad x \cdot 1 = x.$$

- **Inverse:** If  $x \in \mathbb{R}$ , then

$$x + (-x) = 0, \quad x \cdot x^{-1} = 1 \text{ (only if } x \neq 0\text{)}.$$

- **Distributivity:** If  $x, y, p \in \mathbb{R}$ , then

$$p \cdot (x + y) = p \cdot x + p \cdot y.$$

**Remark.** For those with some familiarity with basic abstract algebra, these laws form the definition of a **field**.

With these properties for the real numbers in mind, we immediately run into trouble. *If  $y$  is any real number not equal to 1, then we cannot make sense of  $y \cdot i$ !* As before, when something does not exist, we make it up. In other words, to resolve this issue, *we define new numbers  $y \cdot i$  for any  $y \in \mathbb{R}$  (except  $1 \cdot i = i$ , to remain consistent with the algebraic laws)*. These are called the **imaginary numbers**.

Next, we ask whether we can add imaginary numbers. The answer is not only *yes*, but also that *there is only one reasonable way to do this!* Indeed, given two imaginary numbers  $y_1i$  and  $y_2i$ , the distributive property mandates that we define

$$y_1i + y_2i = (y_1 + y_2)i.$$

Similarly, by the associative property, there is only one reasonable way to multiply a real number and an imaginary number:

$$x \cdot yi = (xy)i.$$

What about multiplying two imaginary numbers? By recalling that  $i$  is defined to be the square root of  $-1$ , we come to the same conclusion:

$$(y_1i) \cdot (y_2i) = (y_1y_2) \cdot (i \cdot i) = -y_1y_2.$$

Now, what about adding a real number to an imaginary number? *Uh oh.* If  $x, y \in \mathbb{R}$ , then  $x + yi$  once again makes no sense. Consequently, we hit the panic button once more and make up even more new numbers: *we go beyond both the real and the imaginary numbers by defining new numbers  $x + yi$  for any  $x, y \in \mathbb{R}$ .*

**Definition 1.3.** These new numbers  $x + yi$ , where  $x, y \in \mathbb{R}$ , are called **complex numbers**. The set of all complex numbers is denoted  $\mathbb{C}$ .

Moreover, given  $x + yi \in \mathbb{C}$ , we define the obvious:

- Its **real part** is  $\operatorname{Re}(x + yi) = x$ .
- Its **imaginary part** is  $\operatorname{Im}(x + yi) = y$ .

Note in particular that any real number is also a complex number. Indeed, if  $x \in \mathbb{R}$ , then we can also write this as  $x + 0i \in \mathbb{C}$ .

At this point, you may be wearily wondering just how many more numbers we would have to make up. *Spoiler alert: this is it, we're finished!* From our definition of the complex numbers, there will be exactly one “reasonable” way to define all the algebraic operations. On top of this, the complex numbers will already be enough to overcome all our issues with taking exponents of negative numbers and with factoring polynomials, though we will not be able to demonstrate this until later.

For now, let us first consider the algebraic operations. Given two complex numbers  $x_1 + y_1i$  and  $x_2 + y_2i$ , the definitions of their sum is

$$(x_1 + y_1i) + (x_2 + y_2i) = (x_1 + x_2) + (y_1 + y_2)i.$$

Keeping in mind the aforementioned algebraic laws, this is the *only* reasonable way to define addition. Similarly, for multiplication, by using the same algebraic laws and the definition  $i \cdot i = -1$ , we arrive again at only one reasonable definition:

$$\begin{aligned} (x_1 + y_1i) \cdot (x_2 + y_2i) &= x_1 \cdot x_2 + x_1 \cdot y_2i + y_1i \cdot x_2 + y_1i \cdot y_2i \\ &= (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)i. \end{aligned}$$

As a result, we can add and multiply any two complex numbers.

**Definition 1.4.** Given  $x_1 + y_1i, x_2 + y_2i \in \mathbb{C}$ , we define

$$\begin{aligned} (x_1 + y_1i) + (x_2 + y_2i) &= (x_1 + x_2) + (y_1 + y_2)i, \\ (x_1 + y_1i) \cdot (x_2 + y_2i) &= (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)i. \end{aligned}$$

**Remark.** Note that for two real numbers, their sum and product as real numbers is the same as their sum and product as complex numbers. Therefore, *the complex number system is an extension of the real number system.*

**Example 1.1.** Compute the following:

$$(1 - 2i) + (-3 + i), \quad (1 + i) \cdot (1 - 2i).$$

*Solution.* There is not much to do, except to compute directly:

$$\begin{aligned} (1 - 2i) + (-3 + i) &= (1 - 3) + (-2 + 1)i = -2 - i, \\ (1 + i) \cdot (1 - 2i) &= [1 \cdot 1 - 1 \cdot (-2)] + [1 \cdot (-2) + 1 \cdot 1]i = 3 - i. \quad \square \end{aligned}$$

If you are sufficiently bored, you can (and should) check the following:

For "fun":

Check that complex addition and multiplication satisfy the commutative, associative, identity, and distributive properties.

In the preceding exercise, the inverse property was deliberately left out, as it merits some additional discussion. First of all, given  $x + yi \in \mathbb{C}$ , it is rather clear that its additive inverse is just  $(-x) + (-y)i$ , since

$$(x + yi) + [(-x) + (-y)i] = (x - x) + (y - y)i = 0.$$

The multiplicative inverse is less obvious but can also be found. First, note that

$$(x + yi) \cdot (x - yi) = (x^2 + y^2) + (yx - xy)i = x^2 + y^2.$$

Dividing both sides by  $x^2 + y^2$  yields

$$(x + yi) \cdot \left( \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2} \cdot i \right) = 1.$$

In other words, any nonzero complex number has a multiplicative inverse, with

$$(1.1) \quad \frac{1}{x + yi} = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2} \cdot i.$$

We can hence conclude that complex addition and multiplication satisfy all the same algebraic properties as were listed for real addition and multiplication. Therefore, *we have successfully extended the real number system to the larger complex number system, in a manner that preserves the algebraic properties.*

**Remark.** On the other hand, extending the trichotomy property for real numbers is a lost cause. For example, it does not make sense to compare, say, 1 and  $i$  to see which is greater. Intuitively, since the complex numbers do not have the geometric structure of a line like the real numbers, there is no natural definition for comparing two arbitrary complex numbers.

**Example 1.2.** Simplify me:

$$\frac{1 + i}{1 - i}.$$

*Solution.* Again, we compute directly, using (1.1):

$$\frac{1 + i}{1 - i} = (1 + i) \cdot \left( \frac{1}{2} + \frac{1}{2}i \right) = i. \quad \square$$

**1.3. The Complex Plane.** Previously, we constructed a square root of  $-1$ , and we created more and more new numbers until we constructed an algebraic system like that for the real numbers. In this process, we made up many things. Thus, we must now face the consequences of our monstrous creation.

We mentioned before that the geometric interpretation of  $\mathbb{R}$  is as a line, hence the name “real line”. We can then ask whether  $\mathbb{C}$  has its own geometric interpretation. We need not work hard to find the answer. Consider the following obvious facts:

- (1) Every complex number  $z$  is characterized by a pair of real numbers, i.e., the real and the imaginary parts of  $z$ .
- (2) Any pair  $(x, y)$  of real numbers generates a unique complex number,  $x + yi$ .

The conclusion from the above is that there is a natural correspondence between the set  $\mathbb{C}$  of complex numbers and the set of all ordered pairs of real numbers, i.e., the **Euclidean plane**. As a result,  $\mathbb{C}$  can be geometrically interpreted as a plane, and as such, we often refer to  $\mathbb{C}$  as the **complex plane**.



Now, you should be familiar with the Euclidean plane and with plane geometry, both from precalculus and from calculus. If not, then you should go review and come back better prepared.

For convenience, we recall the following standard notation:

**Definition 1.5.** We denote the set of all ordered pairs of real numbers by

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}.$$

Recall that  $\mathbb{R}^2$  corresponds to the Euclidean plane.

More specifically, a complex number  $x + yi$  is naturally identified with the point  $(x, y) \in \mathbb{R}^2$ . Consequently, we can visually represent  $\mathbb{C}$  as a plane: given any  $x + yi \in \mathbb{C}$ , we let its real part  $x$  be its horizontal coordinate, and we let its imaginary part be its vertical coordinate. Thus,  $\mathbb{C}$  “looks just like”  $\mathbb{R}^2$ .

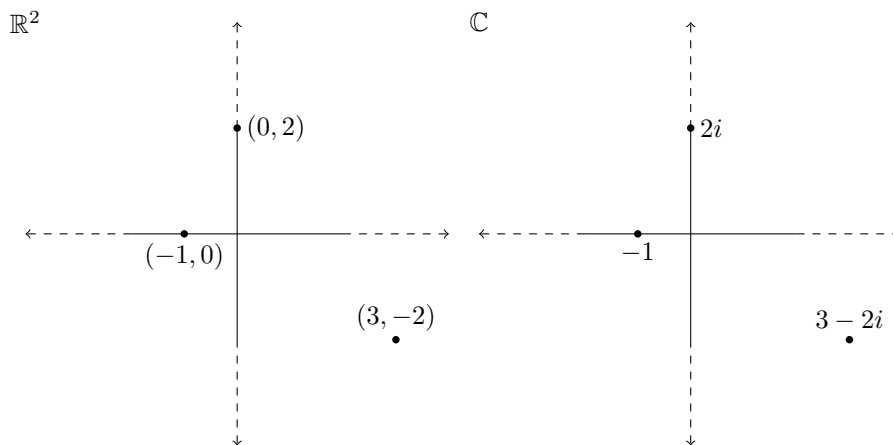


FIGURE 3. Euclidean plane vs. complex plane... they look the same.



The  $x$ -axis, consisting of all the real numbers, is called the **real axis**, while the  $y$ -axis, consisting of all imaginary numbers, is called the **imaginary axis**.

Furthermore, we see that addition of complex numbers corresponds precisely to vector addition in  $\mathbb{R}^2$ . Indeed, given complex numbers  $x_1 + y_1i$  and  $x_2 + y_2i$ , which identify with the points  $(x_1, y_1)$  and  $(x_2, y_2)$ , then their complex sum is

$$(x_1 + y_1i) + (x_2 + y_2i) = (x_1 + x_2) + (y_1 + y_2)i,$$

which corresponds to the point  $(x_1 + x_2, y_1 + y_2) \in \mathbb{R}^2$ , i.e., the vector sum of  $(x_1, y_1)$  and  $(x_2, y_2)$ . Similarly, multiplication in  $\mathbb{C}$  by a *real* number corresponds to scalar multiplication of vectors: given  $c \in \mathbb{R}$ , then

$$c \cdot (x_1 + y_1i) = (cx_1) + (cy_1)i \sim (cx_1, cy_1) = c \cdot (x_1, y_1).$$

To summarize,  $\mathbb{C}$  has the same *linear algebraic*, or *vector*, structure as  $\mathbb{R}^2$ .

So at this point, you may be quite disappointed:

*What?! After all this pulling new numbers out of thin air, all we got was a stupid plane? We did planes and plane vectors in high school already!*

The question now is whether we actually created anything truly new, because what we have thus far seems redundant. However, there is one new concept we created that was not present in Euclidean plane geometry: *multiplication of complex numbers*. Indeed, in the Euclidean picture, one could not meaningfully multiply two points on the plane together to produce a new point. As we shall soon see, there is so much structure to be discovered in this novel operation.

Before we begin exploring complex multiplication in greater detail, though, let us first discuss one more preliminary concept from plane geometry. Recall that any point  $P \in \mathbb{R}^2$  can also be described by its **polar coordinates**,  $(r, \theta)$ , where:

- $r$  is the distance between the origin and  $P$ .
- $\theta$  is the counterclockwise angle that the line segment between the origin and  $P$  makes with the positive  $x$ -axis.

Recall that if  $P$  has polar coordinates  $(r, \theta)$ , then its Cartesian coordinates are

$$(x, y) = (r \cos \theta, r \sin \theta) = r \cdot (\cos \theta, \sin \theta).$$

Similarly, if  $P$  has Cartesian coordinates  $(x, y)$ , then its polar coordinates are

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1} \left( \frac{y}{x} \right).$$

Note in particular that *the angular coordinate  $\theta$  is not unique*. If  $\theta$  is an angular value for  $P$ , then so is  $\theta$  plus any integer multiple of  $2\pi$ . Moreover, *if  $P$  has polar coordinates  $(r, \theta)$ , then all the possible polar coordinates for  $P$  are given by*

$$(r, \theta + 2\pi n), \quad n \in \mathbb{Z}.$$

In summary, the radial coordinate is unique, while the angular coordinate is unique only modulo  $2\pi$ . Some foreshadowing: this lack of uniqueness for the angular coordinate will be the source of many headaches in later discussions.

Now, since the complex plane is essentially the same as the Euclidean plane, this notion of polar coordinates carries over directly to the complex picture. However, since the *complex* plane is philosophically somewhat different from the *Euclidean* plane, we must invent a whole new system of terminology, even if they describe the same things. (Disclaimer: This is not my idea, but the different terminologies are traditional, hence nearly impossible to change.)

Corresponding to the radial coordinate  $r$ , we define the *modulus*:

**Definition 1.6.** Given  $x + yi \in \mathbb{C}$ , we define its **modulus**,  $|x + yi|$ , to be

$$|x + yi| = \sqrt{x^2 + y^2} \in \mathbb{R}.$$

Thinking back to the Euclidean picture, we see that the *modulus*  $|x + yi|$  represents the distance between the origin and the point  $x + yi$  in the complex plane.

The “quantity corresponding to the angular coordinate in the complex plane that really does not need another name but has one anyway” is called the *argument*:

**Definition 1.7.** Given a nonzero complex number  $x + yi$ , its **argument**, denoted  $\arg(x + yi)$ , is the set of all angular coordinate values for the point  $(x, y)$  in the Euclidean plane. This can be cleanly stated as

$$\arg(x + yi) = \left\{ \theta \in \mathbb{R} \mid \tan \theta = \frac{y}{x} \right\}.$$

Note  $\arg(x + yi)$  always contains infinitely many values—in fact,  $\theta \in \arg(x + yi)$  if and only if  $\theta + 2\pi n \in \arg(x + yi)$  for any integer  $n$ . Thus, to summarize, given a complex number  $z = x + yi$ , its polar coordinates are given by

$$(|z|, \theta), \quad \theta \in \arg z.$$

Recalling the relation between Cartesian and polar coordinates, we can hence write

$$z = |z|(\cos \theta + \sin \theta \cdot i), \quad \theta \in \arg z.$$

This is the general polar representation of a complex number.

**Remark.** The argument of zero is not defined. In fact, for  $z = 0$ , any angular value would suffice in the above polar representation.

If you were to start playing around with polar coordinates on the complex plane, then you would very quickly tire of writing “ $\cos \theta + \sin \theta \cdot i$ ” everywhere. To avoid such irritation, we invent an abbreviation for this: *we define the notation*

$$(1.2) \quad e^{\theta i} := \cos \theta + \sin \theta \cdot i.$$

But, this should raise some eyebrows, as well as some objections:

*The symbol  $e^A$  is already reserved for the exponential function! How could you reuse it so carelessly? Have you no shame?!*

If you currently have these (very valid) complaints, rest assured that the symbol “ $e^{\theta i}$ ” is used for very good reason. This formula (1.2), which we have taken for now as just a notational convention, actually contains a deep relationship between the exponential and the trigonometric functions. Moreover, this relation is deep enough such that (1.2) has a name: **Euler’s formula** (named after the famous Swiss mathematician from the 1700’s, Leonhard Euler). While there is an important discussion to be had regarding (1.2), let us defer this until later, as to not distract too much from our current thread on the geometry of the complex plane and the problem of taking powers of negative numbers (hope you didn’t forget about this already!). For now, let us be content with the following:

I.O.U.

You are entitled (later, not now) to a far better explanation of Euler's formula, as more than a definition to memorize.

Therefore, let us conclude by summarizing that if  $z \in \mathbb{C}$  has polar coordinates  $(r, \theta)$ , then its polar representation in the complex plane is

$$z = re^{\theta i} = |z|e^{\theta i}, \quad \theta \in \arg z.$$

**Example 1.3.** Find all polar representations for  $-1 - i$ .

*Proof.* The key is to remember your 45-45-90 triangles (a side note: there really is no particular point to memorizing special triangles, except so that textbook and test writers will always have a set of questions to use that have simple and exact answers). For the radial coordinate, we compute the modulus:

$$|-1 - i| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}.$$

Next, note the points 0,  $-1$ , and  $-1 - i$  form a 45-45-90 triangle, that is, the line segment from the origin to  $-1 - i$  makes a  $45^\circ$  (or  $\pi/4$ ) angle with the negative real axis. As a result, one angular value of  $-1 - i$  is  $\pi + \pi/4 = 5\pi/4$ , so that

$$\arg(-1 - i) = \left\{ \frac{5\pi}{4} + 2\pi n \mid n \in \mathbb{Z} \right\}.$$

Combining the above, we obtain the desired answer:

$$-1 - i = \sqrt{2}e^{(\frac{5\pi}{4} + 2\pi n)i}, \quad n \in \mathbb{Z}. \quad \square$$

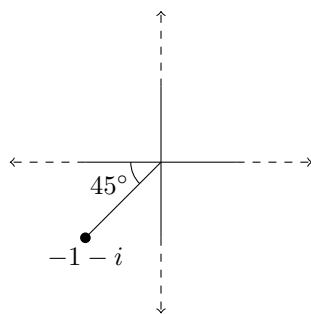


FIGURE 4. Polar representation of  $-1 - i$ .

**1.4. Multiplication and Roots.** We had previously mentioned that addition of complex numbers was equivalent to vector addition in the Euclidean plane and that complex multiplication was a new operation not present in Euclidean geometry. Now that we have some background on polar coordinates, we can explore how multiplication of complex numbers can be interpreted geometrically.

Consider two complex numbers  $z_1$  and  $z_2$ , with polar representations

$$z_1 = r_1 e^{\theta_1 i}, \quad z_2 = r_2 e^{\theta_2 i}.$$

If we multiply them together, then a computation yields

$$\begin{aligned} z_1 z_2 &= r_1(\cos \theta_1 + \sin \theta_1 \cdot i) \cdot r_2(\cos \theta_2 + \sin \theta_2 \cdot i) \\ &= r_1 r_2(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + r_1 r_2(\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2)i. \end{aligned}$$

While this may look like a huge mess at first glance, if you happen to remember your basic trigonometric identities (you shouldn't—this is what the Internet is for), then the *angle addition formulas* state that

$$\begin{aligned} \sin(\theta_1 + \theta_2) &= \cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2, \\ \cos(\theta_1 + \theta_2) &= \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2. \end{aligned}$$

*Bingo!* Just what we need! Plugging this into the previous formula yields

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + \sin(\theta_1 + \theta_2) \cdot i] = (r_1 r_2)e^{(\theta_1 + \theta_2)i}.$$

In particular, the above is a polar representation of  $z_1 z_2$ .

Let now us observe what we have done. If we multiply complex numbers  $z_1$  and  $z_2$ , then the product  $z_1 z_2$  is given as follows:

- The moduli of  $z_1$  and  $z_2$  is the product of the moduli of  $z_1$  and  $z_2$ .
- One argument for  $z_1 z_2$  is the sum of the of arguments for  $z_1$  and  $z_2$ .

The first point above can be summarized as

$$(1.3) \quad |z_1 z_2| = |z_1| |z_2|.$$

The second point is a bit more difficult to pack into a neat formula, since angular values are not unique. However, what we know is the following: *if we are given one angular value for  $z_1$  and one angular value for  $z_2$ , then one angular value for  $z_1 z_2$  can be obtained by summing the above two angular values.* More than this, if we want *all* the angular values of  $z_1 z_2$ , we can obtain them by taking all possible sums of an angular value of  $z_1$  with an angular value of  $z_2$ . For those who really insist on having a neat formula, we can give one in terms of arguments:

$$(1.4) \quad \arg(z_1 z_2) = \arg z_1 + \arg z_2 = \{\theta_1 + \theta_2 \mid \theta_1 \in \arg z_1 \text{ and } \theta_2 \in \arg z_2\}.$$

**Remark.** To obtain  $\arg(z_1 z_2)$ , we could obtain one angular value  $a$  for  $z_1 z_2$ , as before, and then take  $a$  plus any integer multiple of  $2\pi$ .

Geometrically speaking, if  $z \in \mathbb{C}$  has polar representation  $z = re^{\theta i}$ , then when you multiply a number by  $z$ , you transform this number in the following way:

- The modulus is scaled by a factor of  $r$ , i.e., the distance of the number from the origin is scaled by a factor of  $r$ .
- The argument is increased by  $\theta$ , i.e., the point on the plane corresponding to the number is rotated counterclockwise about the origin by angle  $\theta$ .

More generally, given a set of points  $A$  in the complex plane, then multiplying each point of  $A$  by  $z$  achieves the following effect:  *$A$  is dilated by a factor of  $r$  and is rotated counterclockwise about the origin by angle  $\theta$ .* Thus, we see that complex multiplication has a very natural interpretation in terms of plane geometry.

Next, with  $z$  as before, we can multiply  $z$  to itself to obtain

$$z^2 = r^2 e^{2\theta i}.$$

We can repeat this process as many times as we like:

$$z^k = r^k e^{k\theta i}, \quad k \text{ is a positive integer.}$$

Now, what about nonpositive powers of  $z$  (assuming  $z \neq 0$  for negative powers)?

- In the case  $k = 0$ , we have that

$$r^0 e^{0\theta i} = 1 \cdot (\cos 0 + \sin 0 \cdot i) = 1 = z^0.$$

- Next, note the above discussion implies that

$$r^{-1} e^{-\theta i} \cdot r e^{\theta i} = (r^{-1} r) e^{(-\theta + \theta)i} = 1.$$

In other words,  $r^{-1} e^{-\theta i}$  is multiplicative inverse of  $z$ , i.e.,  $z^{-1}$ .

- Finally, for higher negative powers, we have that

$$z^{-k} = (z^{-1})^k = r^{-k} e^{-k\theta i}.$$

Combining all these points, we obtain a convenient way to compute the integer power of a complex number of  $z$  using polar representations: for any integer  $n$ ,

$$(1.5) \quad z = r e^{\theta i} \quad \Rightarrow \quad z^n = r^n e^{n\theta i}.$$

When  $r = 1$ , equation (1.5) is called **de Moivre's formula** (after French mathematician Abraham de Moivre), in case you don't have enough names to remember.

Taking the insight we have just gained, we can now work backwards. Given  $z = r e^{\theta i} \in \mathbb{C}$  and a positive integer  $n$ , *can we find an  $n$ -th root  $z^{1/n}$  of  $z$* , that is, can we find  $w \in \mathbb{C}$  such that  $w^n = z$ ? To find this, we reverse the process from taking the  $n$ -th power: *we take the  $n$ -th root of the modulus, and we divide the argument by  $n$* . With this in mind, if we define

$$w = r^{\frac{1}{n}} e^{\frac{\theta}{n} \cdot i},$$

then from what we have done before,

$$w^n = (r^{\frac{1}{n}})^n e^{n \cdot \frac{\theta}{n} \cdot i} = r e^{\theta i} = z,$$

so that  $w$  is indeed an  $n$ -th root of  $z$ .

Before you pat yourself on the back, we should ask whether we have found *all* the  $n$ -th roots of  $z$ . Unfortunately, the answer is no; for example, 1 has two square roots, 1 and  $-1$ . To find what we missed, we must again deal with the fact that angular values are not unique (as you can see, this is already a pain).

For our more careful analysis, suppose  $w = \rho e^{\sigma i}$  is an  $n$ -th root of  $z$ , so that

$$z = w^n = \rho^n e^{n\sigma i}.$$

Matching moduli, we see that  $|z| = r = \rho^n$ , i.e., that

$$\rho = |w| = r^{\frac{1}{n}}.$$

Next, recall that all the possible angular values for  $z$  are given by

$$\theta + 2\pi k, \quad k \in \mathbb{Z},$$

so that by matching arguments, we obtain the equation  $n\sigma = \theta + 2\pi k$ . Solving this equation for  $\sigma$  yields all the possible angular values of  $w$ :

$$\sigma = \frac{\theta}{n} + \frac{2\pi k}{n}, \quad k \in \mathbb{Z}.$$

Putting it all together, we see that (all) the  $n$ -th roots of  $z$  are given by

$$w_k = r^{\frac{1}{n}} e^{\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)i}, \quad k \in \mathbb{Z}.$$

Thus, we once again have infinitely many polar representations. The question, however, is how many unique numbers are there?

- We begin by taking  $w_0$ , which is one  $n$ -th root of  $z$ .
- Next, note that  $w_1$  is just  $w_0$ , rotated counterclockwise about the origin by angle  $2\pi/n$ , i.e., an  $n$ -th of a full revolution.
- In general,  $w_k$  is just  $w_0$  rotated counterclockwise by angle  $2\pi k/n$ .

Thus, for  $k = 0, 1, \dots, n-1$ , we obtain partial revolutions of  $w_0$  about the origin, each of which yields a unique  $n$ -th root of  $z$ . However, once  $k = n$ , we have gone one full revolution around the origin, and we are back at  $w_0$ . More explicitly,

$$w_n = r^{\frac{1}{n}} e^{\frac{\theta+2\pi n}{n} \cdot i} = r^{\frac{1}{n}} e^{(\frac{\theta}{n}+2\pi)i} = r^{\frac{1}{n}} e^{\frac{\theta}{n} \cdot i} = w_0.$$

Thus, at and beyond  $k = n$ , the  $w_k$ 's begin to repeat themselves, and we no longer have any new values. Combining all our hard work, we conclude the following:

*As long as  $z \neq 0$ , there are exactly  $n$  unique  $n$ -th roots of  $z$ , given by*

$$(1.6) \quad w_k = r^{\frac{1}{n}} e^{\frac{\theta+2\pi k}{n} \cdot i}, \quad k = 0, 1, \dots, n-1.$$

In particular, letting  $z$  be any negative real number, we can now find all the  $n$ -th roots of  $z$ . Therefore, by extending from the real number system to the complex number system, we have now gained the power (pun not intended) to take integer roots of any negative real number. Note that such roots, of course, may not be real numbers. This answers one of the main questions we posed at the beginning of this section. Even better, we can take integer roots of not just real numbers, but any complex number! In other words, *by extending to the complex number system, we now have the ability to always take roots of numbers.*

**Remark.** At the moment, the question of whether we can take *arbitrary real* powers of numbers is still unresolved. We will see in the subsequent section that this will indeed be doable in the complex number system.

We close with some basic examples. We start with the most basic nonreal example, with which we began our discussions: the square roots of  $-1$ . Note that  $-1$ , corresponding to the point  $(-1, 0) \in \mathbb{R}^2$ , has polar coordinates  $(1, \pi)$ . Thus,

$$-1 = 1 \cdot e^{\pi i}.$$

Using our nifty formula (1.6), we see that the square roots of  $-1$  are

$$1^{\frac{1}{2}} e^{\frac{\pi}{2} \cdot i} = e^{\frac{\pi}{2} \cdot i}, \quad 1^{\frac{1}{2}} e^{(\frac{\pi}{2} + \frac{2\pi}{2})i} = e^{\frac{3\pi}{2} \cdot i}.$$

We can state these points in a much simpler fashion:

$$e^{\frac{\pi}{2} \cdot i} = \cos \frac{\pi}{2} + \sin \frac{\pi}{2} \cdot i = i, \quad e^{\frac{3\pi}{2} \cdot i} = \cos \frac{3\pi}{2} + \sin \frac{3\pi}{2} \cdot i = -i.$$

As expected, the square roots of  $-1$  are just  $\pm i$ .

**Example 1.4.** Find all fifth roots of  $-1 + i$ .

*Solution.* Since one polar representation of  $-1 + i$  is

$$-1 + i = \sqrt{2} e^{\frac{3\pi}{4} \cdot i},$$

then (1.6) implies that the fifth roots of  $-1 + i$  are given by

$$\sqrt[5]{2} e^{i(\frac{3\pi}{4} + \frac{2\pi k}{5})} = 2^{\frac{1}{10}} e^{i(\frac{3\pi}{4} + \frac{2\pi k}{5})}. \quad \square$$

Finally, let us consider the  $n$ -th roots of  $1 = 1 \cdot e^{0i}$ , often called the  $n$ -th **roots of unity** (because “unity” is a fancy word for 1). By (1.6), these are given by

$$e_k = e^{\frac{2\pi k}{n}i}, \quad k = 0, 1, \dots, k-1.$$

Geometrically speaking,  $e_0$  is just 1, while  $e_k$  is  $e_0 = 1$  rotated counterclockwise about the origin by angle  $2\pi k/n$  (that is,  $k/n$  of a full revolution). Even more explicitly, the  $n$ -th roots  $e_0 = 1, e_1, \dots, e_{n-1}$  of unity are simply  $n$  points on the unit circle about the origin, evenly spaced apart.

**1.5. Applications and Connections.** Here, we discuss some connections between complex numbers and other mathematical areas. In the first example, we see how complex numbers can be used to better understand some basic ordinary differential equations. In the remaining example, we use some basic concepts in linear algebra to better connect  $\mathbb{C}$  with  $\mathbb{R}^2$ .

1.5.1. *A Connection with Differential Equations.* Let  $c$  be any real constant, and consider the following linear ordinary differential equation:

$$(1.7) \quad y'' = cy.$$

In other words, we want to find a function  $y$ , of one variable, which satisfies (1.7).

If you are familiar with linear differential equations, you would be aware of general methods to solve (1.7). For now, though, let us be less sophisticated; equation (1.7) is simple enough that we can “guess” solutions:

- If  $c > 0$ , then one can quickly check that the following are solutions of (1.7):

$$y_{\pm}(x) = e^{\pm\sqrt{c} \cdot x}.$$

In fact, *every solution of (1.7) is a linear combination of  $y_+$  and  $y_-$ .*

- On the other hand, if  $c < 0$ , then the following are solutions of (1.7):

$$y_1(x) = \cos(\sqrt{|c|} \cdot x), \quad y_2(x) = \sin(\sqrt{|c|} \cdot x).$$

Moreover, *every solution is a linear combination of  $y_1$  and  $y_2$ .*

At first glance, this seems like two rather separate phenomena: the  $c > 0$  case, with exponential functions as solutions, and the  $c < 0$  case, with trigonometric functions as solutions. But, are there connections to be found between these cases?

Let us begin our investigation by considering functions of the form  $y(x) = e^{zx}$ , for some *complex* constant  $z$ . At this point, we have not even made sense of what  $e^{zx}$  means yet (this will be done in later chapters), but for the sake of informal discussion, let us not worry about this here. In addition, let us assume that  $e^{zx}$  satisfies the same differentiation laws as the usual real exponential functions, namely,

$$\frac{d}{dx} e^{zx} = z \cdot e^{zx}.$$

Note when we differentiate this function, we recover a constant times the function itself. As a result,  $y$  is a natural candidate as a solution to (1.7), for any  $c$ .

With our “guess” for  $y$  as above, we see that

$$y''(x) = z^2 \cdot e^{zx} = z^2 \cdot y(x).$$

Thus, for  $y$  to solve (1.7), we require that  $z^2 = c$ , i.e., that  $z$  is a square root of  $c$ . If  $c$  is positive, this implies that  $z = \pm\sqrt{c}$ , and hence  $y = y_{\pm}$ . This is consistent with our previous (rigorous) computations, and all is right with the world.

Now, if  $c$  is negative, then taking its square roots, we have that

$$z = \pm\sqrt{|c|}i.$$

Although we still have not made sense of  $e^{zx}$  for general  $z \in \mathbb{C}$ , we have made sense of them for *imaginary*  $z$ , via Euler's formula, (1.2). Thus, our heuristic discussion indicates that to solve (1.7), we need  $y$  to be one of the following:

$$y^{\pm}(x) = e^{\pm\sqrt{|c|}i \cdot x} = \cos(\sqrt{|c|} \cdot x) \pm i \sin(\sqrt{|c|} \cdot x).$$

You may find the thought of complex-valued solutions a bit unappealing. To recover real-valued solutions, we observe that since  $y^{\pm}$  are solutions to (1.7), so is any linear combination of  $y^+$  and  $y^-$ . In particular, the following are solutions of (1.7):

$$\frac{1}{2}[y^+(x) + y^-(x)] = \cos(\sqrt{|c|} \cdot x), \quad \frac{1}{2i}[y^+(x) - y^-(x)] = \sin(\sqrt{|c|} \cdot x).$$

We have arrived at the actual general solutions to (1.7)!

Of course, what we have done thus far was mathematical bullshit, since we never even defined  $e^{zx}$ , and since we have not discussed differentiation of complex-valued functions. Basically, we groped around in the dark until we reached the actual solutions we wanted. However, the above discussion suggests that perhaps we *can* make sense of  $e^{zx}$ , and that *if we were to do this correctly, then the above discussion could be made completely rigorous*. A major spoiler is that this is in fact true—we will make all of the above rigorous in the upcoming chapters. In fact, this discussion is the general method for solving equations such as (1.7).

Furthermore, (1.7) suggests a relation between exponential and trigonometric functions. It also suggests that Euler's formula is far deeper than just a definition.

**1.5.2. A Connection with (Real-)Linear Algebra.** Previously, we mentioned that  $\mathbb{C}$  and  $\mathbb{R}^2$  shared the same linear structure—that complex addition on  $\mathbb{C}$  is equivalent to vector addition on  $\mathbb{R}^2$ , and that multiplication by a real number on  $\mathbb{C}$  is equivalent to scalar multiplication on  $\mathbb{R}^2$ . Thus, our study of basic algebraic operations on the complex numbers includes the contents of (real-)linear algebra on  $\mathbb{R}^2$ .

We also mentioned that complex multiplication is a new operation that has no parallel in Euclidean geometry, and that this operation is the source of much of the new structure to be found in complex analysis. On the other hand, we can make a connection between complex multiplication and linear algebra in two dimensions. To be more specific, *multiplication by a complex number is a linear transformation*, in the same sense as you learned in linear algebra.

We start with the simplest case: multiplication of a complex number by  $i$ . Consider an arbitrary  $z = (x + yi) \in \mathbb{C}$ , which we can write as a column vector,

$$z = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Now, if we multiply  $z$  by  $i$  and then write this as a column vector, we obtain

$$iz = -y + xi \simeq \begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$



In other words, multiplication by  $i$  on the complex side is equivalent to multiplication on the real side by the  $2 \times 2$  matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Furthermore, multiplication by a complex number  $w = a + bi \in \mathbb{C}$  also corresponds to a matrix multiplication on  $\mathbb{R}^2$ . More specifically, with  $z$  as before,

$$wz = (ax - by) + (ay + bx)i \simeq \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

In other words, complex multiplication can be interpreted on the real side as a linear transformation on  $\mathbb{R}^2$ , with the above matrix representation. This observation will be crucial in future discussions when we relate real and complex differentiation, in particular in deriving the role of the Cauchy-Riemann equations.

#### REFERENCES

1. S. Fisher, *Complex variables*, 2 ed., Dover Publications, 1999.