8 Nonlinear Wave Equations: Vector Field Methods, Global and Long-time Existence

Thus far, we have, through Theorem 7.2, established local existence and uniqueness for the quadratic derivative nonlinear wave equation (7.3). One consequence of this result is the notion of maximal solution (see Corollary 7.6), as well as a basic understanding of what must happen if such a solution breaks down in finite time (see (7.22) and Corollary 7.7).

What is not yet clear, however, is whether there actually exist solutions that break down in finite time. Unfortunately, explicit "blow-up" solutions can be easily constructed. For example, consider the following special case of (7.3):

$$\Box\phi = (\partial_t\phi)^2, \qquad \phi|_{t=0} = \phi_0, \qquad \partial_t\phi|_{t=0} = \phi_1. \tag{8.1}$$

If we assume that ϕ depends only on t, and we set $y := \partial_t$, then (8.1) becomes

$$y' = -y^2.$$

This is now an ODE that can be solved explicitly:[43]

$$\partial_t\phi(t) = y(t) = \frac{1}{t + \frac{1}{C}}, \qquad \phi_1 = y(0) = C, \qquad C \in \mathbb{R} \setminus \{0\}.$$

In particular, if $C < 0$, then $\partial_t\phi$ (and also ϕ) blows up at finite time $T_+ = |C|^{-1}$.

One can object to the above "counterexample" because the initial data ϕ_1, a constant function, fails to lie in $H^s(\mathbb{R}^n)$. However, this shortcoming can be addressed using the local uniqueness property of Corollary 7.8 (and the remark following its proof). Indeed, suppose one alters ϕ_1 so that it remains a negative constant function on a large enough ball $B_0(R)$, but then smoothly transitions to the zero function outside a larger ball $B_0(R+1)$. Then, finite speed of propagation implies that within the cone

$$\mathcal{C} := \{(t,x) \mid |x| \leq R - |t|\},$$

the new solution ϕ is identical to ODE solution. As a result, as long as R is large enough, this new ϕ will see the same blowup behaviour that was constructed in the ODE setting.

On the other hand, one can still ask whether global existence may hold for *sufficiently small* initial data, for which the linear behaviour is expected to dominate for long times. The main result of this chapter is an affirmative answer for sufficiently high dimensions:

Theorem 8.1 (Small-data global and long-time existence, [Klai1985]). *Consider the initial value problem*

$$\Box\phi = Q(\partial\phi, \partial\phi), \qquad \phi|_{t=0} = \varepsilon\phi_0, \qquad \partial_t\phi|_{t=0} = \varepsilon\phi_1, \tag{8.2}$$

where $\varepsilon > 0$, and where the profiles of the initial data satisfy $\phi_0, \phi_1 \in \mathcal{S}(\mathbb{R}^n)$. Suppose in addition that ε is sufficiently small, with respect to n, ϕ_0, and ϕ_1:

- *If $n \geq 4$, then (8.2) has a unique global solution.*
- *Otherwise, letting $|T_\pm|$ be as in Corollary 7.6, the maximal solution ϕ to (8.2) satisfies*
  - *If $n = 3$, then $|T_\pm| \geq e^{C\varepsilon^{-1}}$.*

[43] There is also the trivial solution $y \equiv 0$.

– *If* $n = 2$, *then* $|T_\pm| \geq C\varepsilon^{-2}$.

– *If* $n = 1$, *then* $|T_\pm| \geq C\varepsilon^{-1}$.

Here, the constants C *depend on the profiles* ϕ_i.

The remainder of this chapter is dedicated to the proof of Theorem 8.1. To keep the exposition brief, we omit some of the more computational and technical elements of the proof; for more detailed treatments, as well as generalisations of Theorem 8.1, the reader is referred to [Selb2001, Ch. 7] or [Horm1997, Sogg2008].

Before this, a few preliminary remarks on the theorem are in order.

Remark 8.2. Since the ϕ_i are assumed to lie in $\mathcal{S}(\mathbb{R}^n)$, the initial data $\varepsilon\phi_i$ lie in every H^s-space. As a result, all the machinery from the local theory applies, and one can speak of maximal solutions of (8.2). Furthermore, since these solution curves lie in every H^s-space, it follows that the maximal solution ϕ is actually a smooth classical solution of (8.2).

Remark 8.3. The uniqueness arguments from Theorem 7.2 also carry over to the current setting. Thus, we only need to concern ourselves with existence here.

Remark 8.4. Note that although small-data global existence is not proved for low dimensions $n < 4$ in Theorem 8.1, one does obtain weaker *long-time existence* results, in the form of lower bounds on the timespan $T_\pm$ of solutions.

8.1 Preliminary Ideas

From now on, we let $\phi : (T_-, T_+) \times \mathbb{R}^n \to \mathbb{R}$ be the maximal solution to (8.2), as obtained from Theorem 7.2 and Corollary 7.6. To prove Theorem 8.1, we must hence show $|T_\pm| = \infty$. Moreover, we focus on showing $T_+ = \infty$, since negative times can be handled analogously.

Recall from the previous chapter that the local theory behind (8.2) revolves around energy-type estimates of the form

$$\mathcal{E}_0(t) \lesssim \mathcal{E}(0) + \int_0^t [\mathcal{E}(\tau)]^2 d\tau, \qquad t \in [0, T_+), \tag{8.3}$$

where the "energy" $\mathcal{E}(t)$ is given in terms of H^s-norms:

$$\mathcal{E}(t) := \|\phi(t)\|_{H^{s+1}} + \|\partial_t \phi(t)\|_{H^s}, \qquad s > \frac{n}{2}. \tag{8.4}$$

In particular, both local existence and uniqueness followed from this type of estimate.

A major guiding intuition was that whenever t is small, the nonlinear $\mathcal{E}^2$-integral in (8.3) will not interfere appreciably with the linear evolution. However, since this intuition breaks down whenever t is large, (8.3) is not enough to ensure $\mathcal{E}(t)$ does not blow up at a finite time. Thus, our local theory, based around (8.3), cannot be sufficient to derive global existence.

Suppose on the other hand that we have a stronger "energy estimate",

$$\mathcal{E}(t) \lesssim \mathcal{E}(0) + \int_0^t \frac{[\mathcal{E}(\tau)]^2}{(1+\tau)^p} d\tau, \qquad p > 0, \tag{8.5}$$

where $\mathcal{E}(t)$ now denotes some alternate "energy quantity". In other words, suppose the nonlinear estimate comes with an additional decay in time. If $p > 1$, and hence $(1+\tau)^{-p}$ is integrable on $[0, \infty)$, then the largeness of t is no longer the devastating obstruction it once was. In this case, the smallness of $\mathcal{E}(t)$ itself is sufficient to show that the nonlinear evolution is dominated by the linear evolution, regardless of the size of t.

Indeed, using the integrability of $(1+\tau)^{-p}$ results in the estimate

$$\sup_{0\leq\tau\leq t}\mathcal{E}(\tau)\leq C\mathcal{E}(0)+C_p\left[\sup_{0\leq\tau\leq t}\mathcal{E}(\tau)\right]^2$$

Using a continuity argument, as described in Section 1.4, one can then uniformly bound $\mathcal{E}(t)$ for all $t\in[0,T_+)$. (In fact, this was essentially demonstrated by the computations in Example 1.16.) This propagation property for the modified energy for all times is the main ingredient to improving from local to global existence when $n\geq 4$.

On the other hand, when $n\leq 3$, the power p that one can obtain will be small enough such that $(1+\tau)^{-p}$ is no longer integrable. In this case, one can no longer obtain global existence, but one can still estimate how large t can be before the nonlinear evolution can dominate. Indeed, the nonlinear effects become non-negligible whenever the integral

$$\int_0^t(1+\tau)^{-p}d\tau$$

becomes large.[44] In fact, this consideration is directly responsible for the lower bounds on the times of existence $|T_\pm|$ in Theorem 8.1 when $n\leq 3$.

In light of the above, the pressing questions are then the following:

1. What is this modified energy quantity $\mathcal{E}(t)$?
2. How does one obtain this improved energy estimate (8.5) for $\mathcal{E}(t)$?

8.2 The Invariant Vector Fields

Recall that the unmodified energy is obtained by taking s derivatives of $\partial\phi$ and measuring the L^2-norm. These derivatives ∂_t and ∇_x are handy in particular because they commute with the wave operator. In fact, one can view the H^s-energy estimate for $\partial\phi$ as the L^2-energy estimate applied to both $\partial\phi$ and "$\partial\nabla_x^s\phi$".

With this in mind, it makes sense to enlarge our set of derivatives to other operators that commute with $\Box$. We do this by defining the following set of vector fields on $\mathbb{R}^{n+1}$:

- *Translations*: The Cartesian coordinate vector fields

$$\partial_0:=\partial_t,\partial_1:=\partial_{x^1},\ldots,\partial_n:=\partial_{x^n},\tag{8.6}$$

which generate the spacetime translations of $\mathbb{R}\times\mathbb{R}^n$.[45]

- *Spatial rotations*: The vector fields,

$$\Omega_{ij}:=x^j\partial_i-x^i\partial_j,\qquad 1\leq i<j\leq n,\tag{8.7}$$

which generate spatial rotations on each level set of t.

- *Lorentz boosts*: The vector fields,

$$\Omega_{0j}:=x^j\partial_t+t\partial_j,\qquad 1\leq i\leq n,\tag{8.8}$$

which generate *Lorentz boosts* on $\mathbb{R}\times\mathbb{R}^n$.

[44] Whenever this integral is not large, one can still bound $\mathcal{E}(t)$ via the above continuity argument.

[45] More specifically, transport along the integral curves of the ∂_α's are precisely translations in $\mathbb{R}\times\mathbb{R}^n$.

- *Scaling/dilation*: The vector field

$$S := t\partial_t + \sum_{i=1}^{n} x^i \partial_i, \tag{8.9}$$

which generates the (spacetime) dilations on $\mathbb{R} \times \mathbb{R}^n$.

Note that (8.6)-(8.9) define exactly

$$\gamma_n := (n+1) + \frac{n(n-1)}{2} + n + 1 = \frac{(n+2)(n+1)}{2} + 1$$

independent vector fields. For future notational convenience, we order these vector fields in some arbitrary manner, and we label them as $\Gamma_1, \dots, \Gamma_{\gamma_n}$.

The main algebraic properties of the Γ_a's are given in the following lemma:

Lemma 8.5. *The scaling vector field satisfies*

$$[\Box, S] := \Box S - S\Box = 2\Box, \tag{8.10}$$

while any other such vector field $\Gamma_a \neq S$ satisfies

$$[\Box, \Gamma_a] := \Box\Gamma_a - \Gamma_a\Box = 0. \tag{8.11}$$

Furthermore, for any Γ_b and Cartesian derivative ∂_α, we have

$$[\partial_\alpha, \Gamma_b] = \sum_{\beta=0}^{n} c^\beta_{\alpha b} \partial_\beta, \qquad c^\beta_{\alpha b} \in \mathbb{R}. \tag{8.12}$$

Proof. These identities can be verified through direct computation. In particular, (8.12) is an consequence of the fact that for any such Γ_a, its coefficients (expressed in Cartesian coordinates) are always either constant or one of the Cartesian coordinate functions. □

We will use multi-index notation to denote successive applications of various such Γ_a's. More specifically, given a multi-index $I = (I_1, \dots, I_d)$, where $1 \leq I_i \leq \gamma_n$, we define

$$\Gamma^I = \Gamma_{I_1}\Gamma_{I_2}\dots\Gamma_{I_d}. \tag{8.13}$$

Note that since the Γ_a's generally do not commute with each other, the ordering of the coefficients in such a multi-index I carries nontrivial information.

8.2.1 Geometric Ideas

The key intuitions behind the vector fields (8.6)-(8.9) are actually geometric in nature. To fully appreciate these ideas, one must invoke some basic notions from differential geometry. We give a brief summary of these observations here.

Remark 8.6. In the context of Theorem 8.1, the properties we will need are the identities in Lemma 8.5, which can be computed without reference to any geometric discussions. Thus, the intuitions discussed here are not essential to the proof of Theorem 8.1.

Recall that *Minkowksi spacetime* can be described as the manifold $(\mathbb{R} \times \mathbb{R}^n, m)$, where m is the *Minkowski metric*, i.e., the symmetric covariant 2-tensor on $\mathbb{R}^{n+1}$ given by

$$m := -(dt)^2 + (dx^1)^2 + \dots + (dx^n)^2.$$

In particular, the Minkowski metric differs from the Euclidean metric on $\mathbb{R}^{1+n}$,

$$e := (dt)^2 + (dx^1)^2 + \cdots + (dx^n)^2,$$

only by a reversal of sign in the t-component. However, this change in sign makes Minkowski geometry radically different from the more familiar Euclidean geometry.

Remark 8.7. Minkowski spacetime is the setting for Einstein's theory of *special relativity.*

Furthermore, the wave operator $\Box$ is intrinsic to Minkowski spacetime. Indeed, $\Box$ is the Laplace-Beltrami operator associated with $(\mathbb{R}^{n+1}, m)$,

$$\Box = m^{ij} D^2_{ij} = -\partial_t^2 + \sum_{i=1}^{n} \partial_i^2,$$

where D is the covariant derivative with respect to m. As a result, one would expect that any symmetry of Minkowski spacetime would behave well with respect to $\Box$.

Observe next that any vector field Γ_a given by (8.6)-(8.8) is a *Killing vector field* on Minkowski spacetime, that is, Γ_a *generates symmetries of Minkowski spacetime.* In differential geometric terms, this is given by the condition $\mathcal{L}_{\Gamma_a} m = 0$, where $\mathcal{L}$ denotes the Lie derivative. In other words, transport along the integral curves of Γ_a does not change the Minkowski metric, hence such a transformation yields a symmetry of Minkowski spacetime. Since $\Box$ arises entirely from Minkowski geometry, transporting along Γ_a also preserves the wave operator. This is the main geometric intuition behind (8.11).

Remark 8.8. In fact, the vector fields (8.6)-(8.8) generate the Lie algebra of all Killing vector fields on Minkowski spacetime.

On the other hand, the scaling vector field S in (8.9) is *not* a Killing vector field and hence does not generate a symmetry of Minkowski spacetime. However, S is a *conformal Killing vector field*, that is, S generates a conformal symmetry of Minkowski spacetime. As this is not a full symmetry, S will not commute with $\Box$, but the conformal symmetry property ensures that this commutator is relatively simple; see (8.10).

8.3 The Modified Energy

Because the vector fields Γ_a commute so well with $\Box$, see (8.10) and (8.11), then $\Gamma_a \phi$ also satisfies a "nice" nonlinear wave equation:

$$\Box \Gamma_a \phi = \Gamma_a \Box \phi + c \Box \phi = \Gamma_a \partial \phi \cdot \partial \phi + c (\partial \phi)^2, \qquad c \in \mathbb{R}. \tag{8.14}$$

As a result, one can also apply energy estimates to control $\partial \Gamma_a \phi$ in terms of the initial data. Moreover, the same observations hold for any number of Γ_a's applied to ϕ—for any multi-index $I = (I_1, \dots, I_d)$, with $1 \leq I_i \leq \gamma_n$, one has

$$|\Box \Gamma^I \phi| \leq |\Gamma^I \Box \phi| + |[\Box, \Gamma^I] \phi| \lesssim \sum_{|J| \leq |I|} |\Gamma^J \Box \phi|, \tag{8.15}$$

where the sum is over all multi-indices $J = (J_1, \dots, J_m)$ with length $|J| = m \leq d = |I|$. Applying (8.2) and then (8.12) to the right-hand side of (8.15), we see that

$$|\Box \Gamma^I \phi| \lesssim \sum_{|J|+|K| \leq |I|} |\Gamma^J \partial \phi| |\Gamma^K \partial \phi| \lesssim \sum_{|J|+|K| \leq |I|} |\partial \Gamma^J \phi| |\partial \Gamma^K \phi|. \tag{8.16}$$

This leads us to define the following modified energy quantity for ϕ:[46]

$$\mathcal{E}(t) = \sum_{|I| \leq n+4} \|\partial \Gamma^I \phi(t)\|_{L^2}. \tag{8.17}$$

We now wish to show that this satisfies an improved energy estimate (8.5).

Applying the linear estimate (5.26) to $\Gamma^I \phi$, with $s = 0$, yields

$$\|\partial \Gamma^I \phi(t)\|_{L^2} \lesssim \|\partial \Gamma^I \phi(0)\|_{L^2} + \int_0^t \|\Box \Gamma^I \phi(\tau)\|_{L^2} d\tau.$$

Summing the above over $|I| \leq n+4$ and applying (8.16) yields

$$\mathcal{E}(t) \lesssim \mathcal{E}(0) + \sum_{|J|+|K| \leq n+4} \int_0^t \| |\partial \Gamma^J \phi(\tau)| |\partial \Gamma^K \phi(\tau)| \|_{L^2} d\tau. \tag{8.18}$$

Now, since $|J|+|K|$ on the right-hand side of (8.18) is at most $n+4$, we can assume without loss of generality that $|J| \leq n/2 + 2$. Using Hölder's inequality results in the bound

$$\begin{aligned} \mathcal{E}(t) &\lesssim \mathcal{E}(0) + \int_0^t \sum_{|J| \leq \frac{n}{2}+2} \|\partial \Gamma^J \phi(\tau)\|_{L^\infty} \sum_{|K| \leq n+4} \|\partial \Gamma^K \phi(\tau)\|_{L^2} d\tau \qquad (8.19) \\ &\lesssim \mathcal{E}(0) + \int_0^t \sum_{|J| \leq \frac{n}{2}+2} \|\partial \Gamma^J \phi(\tau)\|_{L^\infty} \cdot \mathcal{E}(\tau) \cdot d\tau. \end{aligned}$$

8.3.1 Sobolev Bounds with Decay

Previously, we controlled L^∞-norms of ϕ by H^s-energies by applying the Sobolev inequality (7.10). In our setting, this results in the crude bound

$$\|\phi(t)\|_{L^\infty} \lesssim \sum_{k \leq \frac{n}{2}+1} \|\partial^k \phi(t)\|_{L^2} \lesssim \sum_{|I| \leq \frac{n}{2}+1} \|\Gamma^I \phi(t)\|_{L^2}. \tag{8.20}$$

Note we are losing a large amount of information here, since we are considering all the vector fields Γ_a, not just the ∂_α's. By leveraging the fact that many of the Γ_a's have growing weights, one sees the possibility of an improvement to (8.20), with additional weights on the left-hand side that grow. In fact, there does exist such an estimate, which is known as the *Klainerman-Sobolev*, or *global Sobolev, inequality*:

Theorem 8.9 (Klainerman-Sobolev inequality). *Let $v \in C^\infty([0,\infty) \times \mathbb{R}^n)$ such that $v(t) \in \mathcal{S}(\mathbb{R}^n)$ for any $t \geq 0$. Then, the following estimate holds for each $t \geq 0$ and $x \in \mathbb{R}^n$:*

$$(1+t+|x|)^{\frac{n-1}{2}} (1+|t-|x||)^{\frac{1}{2}} |v(t,x)| \lesssim \sum_{|I| \leq \frac{n}{2}+1} \|\Gamma^I v(t)\|_{L^2}. \tag{8.21}$$

Roughly, the main idea behind the proof of (8.21) is to write ∂_α as linear combinations of the Γ_a's, which introduce decaying weights. This can be expressed in multiple ways, with each resulting in different weights in time and space. One then applies standard Sobolev inequalities (either on $\mathbb{R}^n$ or on $\mathbb{S}^{n-1}$) and uses the aforementioned algebraic relations to pick up decaying weights. Moreover, depending on the relative sizes of t and $|x|$, one can choose the specific relations and estimates to maximise the decay in the weight. For details, the reader is referred to either [Selb2001, Ch. 7] or [Horm1997, Sogg2008].

[46] Recall again that $\partial := (\partial_t, \nabla_x)$ denotes the spacetime gradient.

Remark 8.10. In the context of Theorem 8.1, the Klainerman-Sobolev estimate suggests decay for ϕ in both t and $|x|$. Furthermore, the weight on the left-hand side of (8.21) indicates that ϕ will decay a half-power *faster* away from the cone $t = |x|$. For our current problem, though, we will not need to consider the decay in $|x|$ or in $|t - |x||$.

In particular, when we apply Theorem 8.9 to (8.19), we obtain

$$\mathcal{E}(t) \lesssim \mathcal{E}(0) + \int_0^t (1+\tau)^{-\frac{n-1}{2}} \sum_{|J|+|K| \leq n+4} \|\Gamma^K \partial \Gamma^J \phi(\tau)\|_{L^2} \cdot \mathcal{E}(\tau) \cdot ds. \tag{8.22}$$

Commuting Γ_a's and ∂_α's using (8.12) yields the following bound:

Lemma 8.11. *For any* $0 \leq t < T_+$,

$$\mathcal{E}(t) \lesssim \mathcal{E}(0) + \int_0^t \frac{[\mathcal{E}(\tau)]^2}{(1+\tau)^{\frac{n-1}{2}}} d\tau. \tag{8.23}$$

Remark 8.12. Note that one must prescribe a high enough number of derivatives in the definition of $\mathcal{E}(t)$, so that after applying (8.21) to the L^∞-factor in (8.19), the resulting L^2-norms are still controlled by $\mathcal{E}(t)$. This is the rationale behind our choice $n + 4$.

8.4 Completion of the Proof

We now apply (8.23) to complete the proof of Theorem 8.1. The main step is the following:

Lemma 8.13. *Assume* ε *in* (8.2) *is sufficiently small. Then:*

- *If* $n = 4$, *then* $\mathcal{E}(t) \lesssim \mathcal{E}(0)$ *for all* $0 \leq t < T_+$.
- *If* $n = 3$, *then* $\mathcal{E}(t) \lesssim \mathcal{E}(0)$ *for all* $0 \leq t < \min(T_+, e^{C\varepsilon^{-1}})$.
- *If* $n = 2$, *then* $\mathcal{E}(t) \lesssim \mathcal{E}(0)$ *for all* $0 \leq t < \min(T_+, C\varepsilon^{-2})$.
- *If* $n = 1$, *then* $\mathcal{E}(t) \lesssim \mathcal{E}(0)$ *for all* $0 \leq t < \min(T_+, C\varepsilon^{-1})$.

Here, C *is a constant depending on* ϕ_0 *and* ϕ_1.

Let us first assume Lemma 8.13 has been established. Applying the standard Sobolev embedding (7.10), we can uniformly bound the spacetime gradient of ϕ:

$$\|\partial\phi(t)\|_{L^\infty} \lesssim \sum_{|I| \lesssim \frac{n}{2}+1} \|\partial \Gamma^I \phi(t)\|_{L^2} \lesssim \mathcal{E}(t). \tag{8.24}$$

When $n \geq 4$, combining Lemma 8.13 and (8.24) results in a uniform bound on $\partial\phi$ on all of $[0, T_+) \times \mathbb{R}^n$. By Corollary 7.7, it follows that $T_+ = \infty$, as desired.

Consider now the case $n = 3$, and suppose $T_+ \leq e^{C\varepsilon^{-1}}$. Again, by Lemma 8.13 and (8.24), one can bound $\partial\phi$ uniformly on $[0, T_+) \times \mathbb{R}^n$. Corollary 7.7 then implies $T_+ = \infty$, resulting in a contradiction. Thus, we conclude $T_+ \geq e^{C\varepsilon^{-1}}$, as desired.

The remaining cases $n < 3$ can be proved in the same manner as for $n = 3$. Thus, to complete the proof of Theorem 8.1, it remains only to prove Lemma 8.13.

8.4.1 The Bootstrap Argument

As mentioned before, the proof of Lemma 8.13 revolves around a continuity argument.[47] For this, we first fix positive constants A and B such that $\mathcal{E}(0) := \varepsilon B \ll \varepsilon A$. Given a time $t \geq 0$, we make the following bootstrap assumption: [48]

BS(t): $\mathcal{E}(t') \leq 2A\varepsilon$ for all $0 \leq t' \leq t$.

The goal then is to derive a strictly better version of **BS**(t).

Suppose first that $n \geq 4$, so that $(1+\tau)^{-\frac{n-1}{2}}$ is integrable on all of $[0,\infty)$. Then, applying (8.23) and the bootstrap assumption **BS**(t), we obtain, for any $0 \leq t' \leq t$,

$$\begin{aligned} \mathcal{E}(t') &\leq C \cdot \mathcal{E}(0) + C \int_0^{t'} \frac{[\mathcal{E}(\tau)]^2}{(1+\tau)^{\frac{n-1}{2}}} d\tau \\ &\leq \varepsilon CB + 4\varepsilon^2 CA^2 \int_0^\infty \frac{1}{(1+\tau)^{\frac{n-1}{2}}} d\tau \\ &\leq \varepsilon CB + \varepsilon^2 C' A^2, \end{aligned} \tag{8.25}$$

where $C' > 0$ is another constant. Note in particular that if ε is sufficiently small, then (8.25) implies a strictly better version of **BS**(t):

$$\mathcal{E}(t') \leq \varepsilon A, \qquad 0 \leq t' \leq t.$$

This implies the desired uniform bound for $\mathcal{E}(t)$ and proves Lemma 8.13 when $n \geq 4$.

Consider next the case $n = 3$. The main idea is that the above bootstrap argument still applies as long as t is not too large. More specifically, assuming **BS**(t) as before, one sees that as long as $t' \leq t \leq e^{C\varepsilon^{-1}}$, the following estimate still holds:

$$\begin{aligned} \mathcal{E}(t') &\leq \varepsilon C' B + 4\varepsilon^2 C' A^2 \int_0^{e^{C\varepsilon^{-1}}} \frac{1}{1+\tau} d\tau \\ &\leq \varepsilon C' B + \varepsilon A \cdot CC'' A. \end{aligned} \tag{8.26}$$

Letting C be small, then one once again obtains a strictly improved version of **BS**(t),

$$\mathcal{E}(t') \leq \varepsilon A, \qquad 0 \leq t' \leq t,$$

as long as $t \leq e^{C\varepsilon^{-1}}$ *for the above* C. A continuity argument (which can be localised to a finite interval) then implies that $\mathcal{E}(t)$ is uniformly bounded for all times $0 \leq t \leq e^{C\varepsilon^{-1}}$.

The proofs of the remaining cases $n < 3$ resemble that of $n = 3$, hence we omit the details here. This completes the proof of Theorem 8.1.

8.5 Additional Remarks

We conclude this chapter with some additional remarks on variants of Theorem 8.1.

8.5.1 Higher-Order Nonlinearities

Theorem 8.1 can be extended to higher-order derivative nonlinearities $\mathcal{N}(\phi, \partial\phi) \approx (\partial\phi)^p$ for $p > 2$. Consider, for concreteness, the cubic derivative nonlinear wave equation

$$\Box\phi = U(\partial\phi, \partial\phi, \partial\phi), \qquad \phi|_{t=0} = \varepsilon\phi_0, \quad \partial_t\phi|_{t=0} = \varepsilon\phi_1, \tag{8.27}$$

[47] For background on continuity arguments, see Section 1.4 and in particular Example 1.16.

[48] Note that the constants A and B depend on the profiles ϕ_0 and ϕ_1.

where U is some trilinear form. Since ϕ is presumed small, then the cubic nonlinearity $(\partial\phi)^3$ should be even smaller than the previous $(\partial\phi)^2$. As a result, one can expect improved small-data global existence results for (8.27).[49]

To be more specific, if we rerun the proof of Theorem 8.1, with $\mathcal{E}$ the modified energy, then the nonlinear term contains two L^∞-factors. This results in the estimate

$$\mathcal{E}(t) \lesssim \mathcal{E}(0) + \int_0^t \frac{[\mathcal{E}(\tau)]^2}{(1+\tau)^{n-1}} d\tau.$$

Since $(1+\tau)^{-(n-1)}$ is integrable when $n \geq 3$, small-data global existence holds for (8.27) whenever $n \geq 3$. Moreover, when $n < 3$, one can again obtain lower bounds on $|T_\pm|$.

This reasoning extends to even higher-order nonlinearities. For instance, for quartic derivative nonlinear wave equations, small-data global existence holds whenever $n \geq 2$.

8.5.2 The Null Condition

Returning to the quadratic case (8.2), small-data global existence now fails for $n = 3$. For example, when $Q(\partial\phi, \partial\phi) = (\partial_t\phi)^2$, every solution with smooth, compactly supported data blows up in finite time; see [John1981]. However, one can still ask whether small-data global existence holds for quadratic nonlinearities containing some special structure.

The key observation is that for such nonlinear waves, not all derivatives of ϕ decay at the same rate; in fact, there are "good" directions which decay better the usual $(1+t)^{-\frac{n-1}{2}}$-rate. For instance, this can be seen in the extra weight $(1 + |t - |x||)^{\frac{1}{2}}$ in the Klainerman-Sobolev inequality, (8.21).[50] As a result, one could possibly expect improved results when Q has the special algebraic property that every term contains at least one "good" component of $\partial\phi$.

The formal expression of this algebraic criterion is known as the *null condition* and was first discovered by Klainerman and Christodoulou; see [Klai1986, Chri1986]. For this, one first defines the *fundamental null forms*:

$$\begin{aligned} Q_0(\partial f, \partial g) &= -\partial_t f \partial_t g + \sum_{i=1}^n \partial_i f \partial_i g, \\ Q_{\alpha\beta}(\partial f, \partial g) &= \partial_\alpha f \partial_\beta g - \partial_\beta f \partial_\alpha g. \end{aligned} \tag{8.28}$$

Then, the null condition is simply that Q is a linear combination of the above forms:

Theorem 8.14. *Let $n = 3$, and suppose Q in* (8.2) *satisfies the above* null condition.[51] *Then, for sufficiently small $\varepsilon > 0$, the solution to* (8.2) *is global.*

We conclude by demonstrating Theorem 8.14 via an example:

Example 8.15. Consider the initial value problem

$$\Box\phi = -Q_0(\partial\phi, \partial\phi), \qquad \phi|_{t=0} = \varepsilon\phi_0, \quad \partial_t\phi|_{t=0} = \varepsilon\phi_1, \tag{8.29}$$

and let $v := e^\phi$. A direct computation shows that v must formally satisfy

$$\Box v = 0, \qquad v|_{t=0} = e^{\phi_0}, \qquad \partial_t v|_{t=0} = \phi_1 e^{\phi_0},$$

which by Theorem 5.3 has a global solution.

[49] The local existence theory of the previous chapter also extends directly to (8.27).
[50] In particular, the proof of Theorem 8.1 did not take advantage of this extra decay.
[51] Note however that $Q_{\alpha\beta}$ can only appear in systems of wave equations.

One can now recover the solution ϕ for (8.29) by reversing the change of variables, $\phi = \log v$. In particular, this solution ϕ exists as long as $v > 0$. A direct computation using (5.9) shows that this indeed holds as long as ϕ_0 and ϕ_1 are small.

References

[Batt1977] Jürgen Batt. Global symmetric solutions of the initial value problem of stellar dynamics. *J. Differ. Equ.*, 25(3):342–364, sep 1977.

[Chri1986] D. Christodoulou. Global solutions of nonlinear hyperbolic equations for small initial data. *Comm. Pure App. Math.*, 39:267–282, 1986.

[Dafe2013] M. Dafermos and I. Rodnianski. Lectures on black holes and linear waves. *Evolution Equations, Clay Mathematics Proceedings (Amer. Math. Soc.)*, 17:97–205, 2013.

[Evan2002] L. C. Evans. *Partial Differential Equations.* AMS, 2002.

[Glassey1986] Robert T. Glassey and Walter A. Strauss. Singularity formation in a collisionless plasma could occur only at high velocities. *Arch. Ration. Mech. Anal.*, 92(1), 1986.

[Glassey1987a] Robert T. Glassey and Walter A. Strauss. Absence of Shocks in an Initially Dilute Collisionless Plasma. *Commun. Math. Phys.*, 113(2):191–208, 1987.

[Glassey1990] Robert T. Glassey and Jack W. Schaeffer. On the 'one and one-half dimensional' relativistic Vlasov-Maxwell system. *Math. Methods Appl. Sci.*, 13(2):169–179, aug 1990.

[Glassey1996] Robert T. Glassey. *The Cauchy Problem in Kinetic Theory.* Society for Industrial and Applied Mathematics, 1996.

[Glassey1997] Robert T. Glassey and Jack W. Schaeffer. The 'Two and One Half Dimensional' Relativistic Vlasov Maxwell System. *Commun. Math. Phys.*, 185(2):257–284, 1997.

[Horm1997] L. Hörmander. *Lectures on Nonlinear Hyperbolic Differential Equations.* Springer-Verlag, 1997.

[Horst1981] E Horst. On the classical solutions of the initial value problem for the unmodified non-linear Vlasov equation I general theory. *Math. Methods Appl. Sci.*, 3(1):229–248, 1981.

[John1981] F. John. Blow-up for quasi-linear wave equations in three space dimensions. *Comm. Pure Appl. Math.*, 34:29–51, 1981.

[Klai1985] S. Klainerman. Uniform decay estimates and the Lorentz invariance of the classical wave equation. *Comm. Pure Appl. Math.*, 38(3):321–332, 1985.

[Klai1986] S. Klainerman. The null condition and global existence to nonlinear wave equations. *Lectures in Applied Mathematics*, 23:293–326, 1986.

[Lions1991] Pierre-Louis Lions and Benoît Perthame. Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system. *Invent. Math.*, 105(1):415–430, dec 1991.

[Pfaffelmoser1992] K Pfaffelmoser. Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data. *J. Differ. Equ.*, 95(2):281–303, feb 1992.

[Rein2007] Gerhard Rein. Collisionless Kinetic Equations from Astrophysics–The Vlasov-Poisson System. *Handb. Differ. Equations Evol. Equations*, 3:383–476, 2007.

[Schaeffer1991] Jack W. Schaeffer. Global existence of smooth solutions to the vlasov poisson system in three dimensions. *Commun. Partial Differ. Equations*, 16(8-9):1313–1335, jan 1991.

[Selb2001] S. Selberg. Lecture notes. Math 632, PDE. http://www.math.ntnu.no/∼sselberg/HopkinsLectures.pdf.

[Sogg2008] C. Sogge. *Lectures on Nonlinear Wave Equations.* International Press, 2008.

[Tao2006] T. Tao. *Nonlinear Dispersive Equations: Local and Global Analysis.* American Mathematical Society, 2006.

[Ukai1978] Seiji Ukai and Takayoshi Okabe. On classical solutions in the large in time of two-dimensional Vlasov's equation. *Osaka J. Math.*, 15:245–261, 1978.