ODE THEORY: C^1-DEPENDENCE ON INITIAL DATA

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In this short note, we give a detailed proof of the C^1-dependence on initial data for a family of solutions of a C^1-system of ODEs. More specifically, we consider the following system,

$$x' = f(t, x), \tag{1}$$

where:

- $t \in \mathbb{R}$ is the independent variable.
- $x : I \to \mathbb{R}^n$ is the unknown, with $I \subseteq \mathbb{R}$ begin some interval containing 0.
- The function $f : \mathbb{R}_t \times \mathbb{R}^n_x \to \mathbb{R}^n$ defines the system of ODEs.

The main statement we wish to prove is the following:

Theorem 1. *Let $J \subseteq \mathbb{R}$ be a nonempty closed interval, and suppose $x : J_r \times [0, T]_t \to \mathbb{R}^n$ is a family of solutions—parametrised by $r \in J$—of (1), that is,*

$$\partial_t x(r, t) = f(t, x(r, t)), \qquad x(r, 0) = x_0(r), \tag{2}$$

with $x_0 : J \to \mathbb{R}^n$ being the corresponding family of initial data.

If f and x_0 are both C^1, then the solution x is also C^1 (in both r and t).

Remark. *Since f is C^1, then f is automatically both locally bounded and locally Lipschitz in x. By the standard existence and uniqueness result, each individual solution $x(r, \cdot)$ exists uniquely and is C^1 in t. Thus, the remaining task is to prove the same C^1-regularity in r.*

C^0-Dependence on Data. As warm-up, we first prove C^0-dependence of the solution x on r. For this, we fix throughout $r, r_0 \in J$, and we set[1]

$$y(t) := y(r, r_0; t) := x(r, t) - x(r_0, t).$$

Differentiating y and recalling (2) results in the identity

$$y'(t) = f(t, x(r, t)) - f(t, x(r_0, t)). \tag{3}$$

Integrating (3) and recalling $y(0) = x_0(r) - x_0(r_0)$, we see that

$$|y(t)| \leq |x_0(r) - x_0(r_0)| + \int_0^t |f(s, x(r, s)) - f(s, x(r_0, s))| ds.$$

Since the domain of x is compact, then by the local Lipschitz properties of f (due to f being C^1),

$$|y(t)| \leq |x_0(r) - x_0(r_0)| + L \int_0^t |x(r, s) - x(r_0, s)| ds = |x_0(r) - x_0(r_0)| + L \int_0^t |y(s)| ds. \tag{4}$$

Applying Gronwall's inequality to (4) yields

$$|x(r, t) - x(r_0, t)| = |y(t)| \leq |x_0(r) - x_0(r_0)| e^{tL}.$$

In particular, as $r \to r_0$, then $x(r, t) \to x(r_0, t)$, so that x is indeed continuous in r for each t.

[1] For simplicity of notation, we suppress the dependence of y on r and r_0.

r-Differentiability of x. The next step is to show that x is differentiable in the r-direction and to determine this derivative. The main idea is to note from (2) that

$$\partial_t \left[\frac{x(r,t) - x(r_0,t)}{r - r_0} \right] = \frac{f(t, x(r,t)) - f(t, x(r_0,t))}{r - r_0}$$

Taking a formal (and completely unrigorous) limit of the above as $r \to r_0$, we can guess that if $\partial_r x$ indeed exists, then it should satisfy the following system of ODEs:[2]

$$\partial_t (\partial_r x)(r_0, t) = \frac{d}{dr} f(t, x(r,t)) \bigg|_{r=r_0} = \nabla_x f(t, x(r_0,t)) \cdot \partial_r x(r_0, t).$$

To be rigorous, we now work backwards: let A denote the solution of the initial value problem[3]

$$\partial_t A(t) = \nabla_x f(t, x(r_0,t)) \cdot A(t), \qquad A(0) = x_0'(r_0). \tag{5}$$

(Since the system is linear, and since $\nabla_x f(t, x(r_0,t))$ is well-defined, $A(t)$ is guaranteed to exist on all of $[0,T]$.) Consider now the function $B : [0,T] \to \mathbb{R}^n$ given by

$$B(t) := \frac{x(r,t) - x(r_0,t)}{r - r_0} - A(t). \tag{6}$$

A direct computation using (5), (6), and the fundamental theorem of calculus yields

$$\begin{aligned} B'(t) &= \frac{1}{r - r_0} \int_0^1 \frac{d}{ds} [f(t, s \cdot x(r,t) + (1-s) \cdot x(r_0,t))] ds - \nabla_x f(t, x(r_0,t)) \cdot A(t) \\ &= \int_0^1 \nabla_x f(t, s \cdot x(r,t) + (1-s) \cdot x(r_0,t)) ds \cdot \frac{x(r,t) - x(r_0,t)}{r - r_0} - \nabla_x f(t, x(r_0,t)) \cdot A(t). \end{aligned}$$

A bit of algebraic manipulation then reveals that

$$B'(t) = C_1(t) \cdot B(t) + C_2(t) \cdot A(t), \tag{7}$$

where

$$\begin{aligned} C_1(t) &= \int_0^1 \nabla_x f(t, s \cdot x(r,t) + (1-s) \cdot x(r_0,t)) ds, \\ C_2(t) &= \int_0^1 [\nabla_x f(t, s \cdot x(r,t) + (1-s) \cdot x(r_0,t)) - \nabla_x f(t, x(r_0,t))] ds. \end{aligned} \tag{8}$$

Again, for notational brevity, we suppress the dependence on r and r_0.

Since x is continuous, and since f is C^1, then both C_1, C_2, and A are uniformly bounded:

$$\sup_{0 \leq t \leq T} [|A(t)| + |C_1(t)| + |C_2(t)|] \leq M. \tag{9}$$

Furthermore, since f is C^1, we can also see that

$$\lim_{r \to r_0} \sup_{0 \leq t \leq T} |C_2(t)| = 0. \tag{10}$$

Thus, integrating (7) and recalling (9) and (10), we obtain the bound

$$\begin{aligned} |B(t)| &\leq |B(0)| + \int_0^t |C_2(s)||A(s)| ds + \int_0^t |C_1(s)||B(s)| ds \\ &\leq |B(0)| + TP \sup_{0 \leq t \leq T} |C_2(t)| + M \int_0^t |B(s)| ds, \end{aligned} \tag{11}$$

[2]Here, the $\nabla_x f$ is an $(n \times n)$-matrix, so that the dot product $\nabla_x f \cdot \partial_r x$ is an n-vector.

[3]Again, we suppress the dependence of A on r and r_0.

and applying the Gronwall inequality to (11) yields

$$|B(t)| \leq e^{tM} \left[|B(0)| + TM \sup_{0 \leq t \leq T} |C_2(t)| \right]. \tag{12}$$

Note that as $r \to r_0$,

$$B(0) = \frac{x_0(r) - x_0(r_0)}{r - r_0} - x_0'(r_0) \to 0. \tag{13}$$

Combining (12) with (10) and (13), we conclude that $B(t) \to 0$ as $r \to r_0$ for each t. By the definition of $B(t)$, this implies that $\partial_r x(r_0, t)$ exists and is equal to $A(t)$.

C^1-Dependence on Data. By showing that $\partial_r x(r_0, t)$ exists and is equal to this $A(t)$ in the preceding argument, we conclude that $\partial_r x$ itself solves a linear system of ODEs:

$$\partial_t(\partial_r x)(r,t) = \nabla_x f(t, x(r,t)) \cdot \partial_r x(r,t), \qquad (\partial_r x)(r, 0) = x_0'(r). \tag{14}$$

This already implies that $\partial_r x$ is continuous with respect to t.

Integrating (14), we obtain the bound

$$\begin{aligned} |\partial_r x(r,t) - \partial_r x(r_0,t)| &\leq |x_0'(r) - x_0'(r_0)| \\ &\quad + \int_0^t |\nabla_x f(s, x(r,s)) \cdot \partial_r x(r,s) - \nabla_x f(s, x(r_0,s)) \cdot \partial_r x(r_0,s)| ds \\ &\leq |x_0'(r) - x_0'(r_0)| + \int_0^t |\nabla_x f(s, x(r,s)) - \nabla_x f(s, x(r_0,s))| |\partial_r x(r,s)| ds \\ &\quad + \int_0^t |\nabla_x f(s, x(r_0,s))| |\partial_r x(r,s) - \partial_r x(r_0,s)| ds. \end{aligned} \tag{15}$$

By Gronwall's inequality and various uniform bounds, (15) implies that

$$|\partial_r x(r,t) - \partial_r x(r_0,t)| \leq e^{Lt} \left[|x_0'(r) - x_0'(r_0)| + C \int_0^t |\nabla_x f(s, x(r,s)) - \nabla_x f(s, x(r_0,s))| ds \right] \tag{16}$$

for some constants $C, L > 0$. Since x_0 and f are C^1, the terms within the brackets on the right-hand side of (16) vanish as $r \to r_0$. This proves the continuity of $\partial_r x$ with respect to r.