δ-DISTRIBUTIONS IN DISPERSIVE EQUATIONS

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1. Introduction

In studying the basic theory of dispersive equations, one encounters various δ-distributions. For example, there is the following well-known fact:

Proposition 1. *Suppose we have a solution to the free Schrödinger equation,*

$$i\partial_t u + \Delta u = 0, \qquad u : \mathbb{R}^{1+n} \to \mathbb{C}, \tag{1}$$

which also satisfies the initial condition

$$u(0) := u|_{t=0} = f, \qquad f : \mathbb{R}^n \to \mathbb{C}. \tag{2}$$

If f is in a sufficiently nice space, say $\mathcal{S}(\mathbb{R}^n)$, then for any $(\tau, \xi) \in \mathbb{R}^{1+n}$,

$$\tilde{u}(\tau, \xi) = \delta(\tau - |\xi|^2)\hat{f}(\xi), \tag{3}$$

where $\hat{f}$ denotes the Fourier transform (in $\mathbb{R}^n$) of f, and where $\tilde{u}$ denotes the spacetime Fourier transform (in $\mathbb{R}^{n+1}$) of u.

Another common fact, for those who have some experience with bilinear estimates, is the following convolution identity:

Proposition 2. *Suppose u and v are solutions of (1), with initial data*

$$u(0) = f, \qquad v(0) = g, \tag{4}$$

as in (2). If f and g are in sufficiently nice spaces, then for any $(\tau, \xi) \in \mathbb{R}^{1+n}$,

$$(\tilde{u} * \tilde{v})(\tau, \xi) = \int_{\mathbb{R}^n} \delta(\tau - |\xi - \eta|^2 - |\eta|^2)\hat{f}(\xi - \eta)\hat{g}(\eta)d\eta. \tag{5}$$

Although much of the dispersive equations literature applies these facts, the details behind their derivations, as well as the precise meanings of right-hand sides of (3) and (5), are often swept under the rug. In this short note, we clarify the definitions behind Propositions 1 and 2, and we prove these propositions in detail.

Remark. *The process we apply here for the Schrödinger equation applies analogously to other dispersive equations, most notably to (half-)wave equations. In the case of half-waves, one generally replaces $\tau - |\xi|^2$ by $\tau \pm |\xi|$.*

Remark. *To prevent from having to deal with pesky factors of 2π from Fourier and inverse Fourier transforms, we replace throughout the usual Euclidean measure μ_m on $\mathbb{R}^m$ by $(2\pi)^{-m/2}\mu_m$. In particular, we set*

$$(2\pi)^{-\frac{1}{2}}d\tau \Rightarrow d\tau, \qquad (2\pi)^{-\frac{n}{2}}d\xi \Rightarrow d\xi,$$

and similarly for dt and dx. [1]

[1] Hat tip to [4, 5] for these convenient tricks.

2. Derivation of Proposition 1

This section is focused on the clarification and proof of (3). While we will make heavy use of distribution theory, in particular for tempered distributions. Throughout, we will remain with the standard nomenclature for these objects:

- Let $\mathcal{S}(\mathbb{R}^m)$ denote the space of smooth, rapidly decaying functions on $\mathbb{R}^m$.
- Let $\mathcal{S}'(\mathbb{R}^m)$ denote the dual space of tempered distributions on $\mathbb{R}^m$.

Remark. *For more on the basic theory of distributions, see, for example,* [3, 4, 5].

2.1. **Pullback Distributions.** Before proving (3), we must first make sense of pullbacks of distributions, in particular over the δ-distribution.

Consider a differentiable function $\psi : \mathbb{R}^{1+n} \to \mathbb{R}$, whose level sets form hypersurfaces of $\mathbb{R}^{1+n}$. [2] Now, if $u \in \mathcal{S}'(\mathbb{R})$, then we wish to make sense of the *pullback* of u over ψ, namely, the composition $u(\psi) \in \mathcal{S}'(\mathbb{R}^{1+n})$.

To motivate this definition, we recall the *coarea formula* (in the smooth case):

Theorem 3 (Coarea formula)**.** *If $F \in \mathcal{S}(\mathbb{R}^{1+n})$, and if ψ is as above, then*

$$\int_{\mathbb{R}^{1+n}} F = \int_{-\infty}^{\infty} \left(\int_{\{\psi = y\}} \frac{F}{|\nabla \psi|} \right) dy, \tag{6}$$

where $\nabla \psi$ is the gradient of ψ, and where the integrals over the level sets $\{\psi = y\}$ are with respect to the induced volume measures.

Proof. In the smooth case here, (6) can be derived using a bit of differential geometry and the change of variables formula; see, for example, [1]. □

Suppose first that u is a nice, smooth function on $\mathbb{R}$. Letting $\varphi \in \mathcal{S}(\mathbb{R}^{1+n})$, we can then apply the coarea formula (6) to $u(\psi)\varphi$, which yields

$$\begin{aligned} \int_{\mathbb{R}^{1+n}} u(\psi)\varphi &= \int_{-\infty}^{\infty} \left(\int_{\{\psi = y\}} \frac{u(\psi)\varphi}{|\nabla \psi|} \right) dy \\ &= \int_{-\infty}^{\infty} u(y) \left(\int_{\{\psi = y\}} \frac{\varphi}{|\nabla \psi|} \right) dy. \end{aligned} \tag{7}$$

This leads to the following definition in the case of distributions:

Definition 4. *If $u \in \mathcal{S}'(\mathbb{R})$, then we define its* pullback *$u(\psi) \in \mathcal{S}'(\mathbb{R}^{1+n})$ by*

$$\langle u(\psi), \varphi \rangle := \left\langle u, y \mapsto \int_{\{\psi = y\}} \frac{\varphi}{|\nabla \psi|} \right\rangle. \tag{8}$$

To be more clear, the right-hand side represents u applied to the test function

$$\varphi^* : \mathbb{R} \to \mathbb{C}, \qquad \varphi^*(y) = \int_{\{\psi = y\}} \frac{\varphi}{|\nabla \psi|}.$$

Recall that the δ-distribution is defined

$$\delta \in \mathcal{S}'(\mathbb{R}), \qquad \langle \delta, \varphi \rangle = \varphi(0). \tag{9}$$

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[2] One is allowed exceptions on sets with some sense of zero measure, though for simplicity, we will ignore this technical point here. This point is relevant, however, for wave equations.

Thus, according to our definition (8), we see that

$$\langle \delta(\psi), \varphi \rangle = \left\langle \delta, y \mapsto \int_{\{\psi = y\}} \frac{\varphi}{|\nabla \psi|} \right\rangle = \int_{\{\psi=0\}} \frac{\varphi}{|\nabla\psi|}. \tag{10}$$

Next, we can further generalize Definition 4 by multiplying by weights:

Definition 5. *Let $u \in \mathcal{S}'(\mathbb{R})$, and let $f : \mathbb{R}^{1+n} \to \mathbb{C}$ be "sufficiently integrable" (depending on u). Then we define $u(\psi)f \in \mathcal{S}'(\mathbb{R}^{1+n})$ by*

$$\langle u(\psi) f, \varphi \rangle := \left\langle u, y \mapsto \int_{\{\psi = y\}} \frac{f\varphi}{|\nabla \psi|} \right\rangle. \tag{11}$$

In particular, with f as before, we have that

$$\langle \delta(\psi) f, \varphi \rangle = \int_{\{\psi=0\}} \frac{f\varphi}{|\nabla\psi|}. \tag{12}$$

2.2. **The Dispersive Relation.** Using Definition 5, we can now make full sense of the right-hand side of (3). [3] In this case, the function ψ is given by

$$\psi(\tau, \xi) = \tau - |\xi|^2, \tag{13}$$

for which the gradient is

$$\nabla\psi(\tau,\xi) = (1, -2\xi), \qquad |\nabla\psi(\tau,\xi)| = \sqrt{1 + 4|\xi|^2}. \tag{14}$$

As a result, given $\varphi \in \mathcal{S}(\mathbb{R}^{1+n})$, we have from (12) and (14) that

$$\begin{aligned} \langle \delta(\tau - |\xi|^2) \hat{f}(\xi), \varphi \rangle &= \int_{\{\tau - |\xi|^2 = 0\}} \frac{\hat{f}(\xi)\varphi(\tau,\xi)}{\sqrt{1+4|\xi|^2}} \\ &= \int_{\{\tau - |\xi|^2 = 0\}} \frac{\hat{f}(\xi)\varphi(|\xi|^2,\xi)}{\sqrt{1+4|\xi|^2}}. \end{aligned} \tag{15}$$

To deal with the integral over the paraboloid $\tau = |\xi|^2$, we note that this hypersurface can be parametrized by the spatial variable $\xi \in \mathbb{R}^n$:

$$\xi \mapsto (h(\xi), \xi) = (|\xi|^2, \xi).$$

Recall from calculus and differential geometry the following relation:

$$\begin{aligned} \int_{\{\tau - |\xi|^2 = 0\}} \frac{\hat{f}(\xi)\varphi(|\xi|^2,\xi)}{\sqrt{1+4|\xi|^2}} &= \int_{\mathbb{R}^n} \frac{\hat{f}(\xi)\varphi(|\xi|^2,\xi)}{\sqrt{1+4|\xi|^2}} \cdot \sqrt{1 + |h(\xi)|} \cdot d\xi \\ &= \int_{\mathbb{R}^n} \hat{f}(\xi)\varphi(|\xi|^2,\xi) d\xi. \end{aligned} \tag{16}$$

From (15) and (16), we obtain an explicit formula for the right-hand side of (3):

$$\langle \delta(\tau - |\xi|^2) \hat{f}(\xi), \varphi \rangle = \int_{\mathbb{R}^n} \hat{f}(\xi)\varphi(|\xi|^2,\xi) d\xi. \tag{17}$$

[3] The "weight" here is a function $F(\tau,\xi)$ that depends only on ξ, i.e., $F(\tau,\xi) = \hat{f}(\xi)$.

2.3. **Completion of the Proof.** To complete the derivation of (1), we must now understand the left-hand side of (1). Given $\varphi \in \mathcal{S}(\mathbb{R}^{1+n})$, we use the distribution definition of Fourier transforms, along with the fact that u is a function, to obtain

$$\langle \tilde{u}, \varphi \rangle = \int_{\mathbb{R}} \int_{\mathbb{R}^n} u(t,x) \tilde{\varphi}(t,x) dx dt \tag{18}$$

Since the Schrödinger evolution preserves L^2-norms, $u(t)$ has *spatial* Fourier transform in $L^2(\mathbb{R}^n)$. Also, $\hat{u}(t)$ is given by the explicit representation formula [4]

$$\hat{u}(t, \xi) = e^{it|\xi|^2} \hat{f}(\xi). \tag{19}$$

Thus, letting $\mathcal{F}_t$ denote the Fourier transform only in time, and focusing on the inner spatial integral on the right-hand side of (18), we see that

$$\begin{aligned} \langle \tilde{u}, \varphi \rangle &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \hat{u}(t,\xi) \cdot (\mathcal{F}_t \varphi)(t, \xi) \cdot d\xi dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{it|\xi|^2} \hat{f}(\xi) \cdot (\mathcal{F}_t \varphi)(t, \xi) \cdot d\xi dt. \end{aligned} \tag{20}$$

Next, we expand $\mathcal{F}_t \varphi$ and apply Fubini's theorem:

$$\langle \tilde{u}, \varphi \rangle = \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}} \int_{\mathbb{R}} e^{-it(\tau - |\xi|^2)} \cdot \hat{f}(\xi) \varphi(\tau, \xi) \cdot d\tau dt \right] d\xi. \tag{21}$$

From here, our desired result follows from the fact that the Fourier transform of the function $t \mapsto e^{iat}$ is $\delta(\cdot - a)$. However, let us compute this explicitly.

For each fixed $\xi \in \mathbb{R}^n$, we apply a change of variables $\tau \mapsto \tau + |\xi|^2$:

$$\begin{aligned} \langle \tilde{u}, \varphi \rangle &= \int_{\mathbb{R}^n} \hat{f}(\xi) \left[\int_{\mathbb{R}} \int_{\mathbb{R}} e^{-it\tau} \varphi(\tau + |\xi|^2, \xi) \cdot d\tau dt \right] d\xi \\ &= \int_{\mathbb{R}^n} \hat{f}(\xi) \left[\int_{\mathbb{R}} \mathcal{F}_t (\varphi_\xi)(t) dt \right] d\xi, \end{aligned} \tag{22}$$

where $\varphi_\xi : \mathbb{R} \to \mathbb{C}$ is defined $\varphi_\xi(\tau) = \varphi(\tau + |\xi|^2, \xi)$. Since the inner integral on the right-hand side of (22) is simply the inverse Fourier transform at 0, we obtain

$$\langle \tilde{u}, \varphi \rangle = \int_{\mathbb{R}^n} \hat{f}(\xi) \varphi_\xi(0) \cdot d\xi = \int_{\mathbb{R}^n} \hat{f}(\xi) \varphi(|\xi|^2, \xi) d\xi. \tag{23}$$

Finally, combining (17) and (23) concludes the derivation of (3).

3. Derivation of Proposition 2

We now turn our attention to the identity (5). For this, the main new ingredient is to recall and understand how convolutions are defined for distributions.

3.1. **Convolutions.** Recall that given $f, g \in \mathcal{S}(\mathbb{R}^m)$, its convolution is defined by

$$f * g \in \mathcal{S}(\mathbb{R}^m), \qquad (f * g)(x) = \int_{\mathbb{R}^m} f(y) g(x - y) dy. \tag{24}$$

Moreover, convolutions are commutative and associative: if $f, g, h \in \mathcal{S}(\mathbb{R}^m)$, then

$$f * g = g * f, \qquad (f * g) * h = f * (g * h). \tag{25}$$

These properties motivate the definition of convolutions of distributions.

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[4]For more details on the basic theory of dispersive equations, see, e.g., [6].

Indeed, given f and g as above, as well as $\varphi \in \mathcal{S}(\mathbb{R}^m)$, we write

$$\int_{\mathbb{R}^m} (f * g)(x)\varphi(x)dx = \int_{\mathbb{R}^m} (f * g)(x)\varphi^\star(-x)dx, \tag{26}$$

where $\varphi^\star \in \mathcal{S}(\mathbb{R}^m)$ is the reflection $\varphi^\star(x) = \varphi(-x)$. Recalling the definition (24), as well as the associative property of convolutions, we see that

$$\int_{\mathbb{R}^m} (f * g)(x)\varphi(x)dx = [(f * g) * \varphi^\star](0) = [f * (g * \varphi^\star)](0). \tag{27}$$

Expanding out the right-hand side, the above becomes

$$\begin{aligned} \int_{\mathbb{R}^m} (f * g)(x)\varphi(x)dx &= \int_{\mathbb{R}^m} f(x)(g * \varphi^\star)(-x)dx \\ &= \int_{\mathbb{R}^m} f(x)\left[\int_{\mathbb{R}^m} g(y)\varphi^\star(-x-y)dy\right] dx \\ &= \int_{\mathbb{R}^m} f(x)\left[\int_{\mathbb{R}^m} g(y)\varphi(x+y)dy\right] dx \end{aligned} \tag{28}$$

Thus, if we define for each $x \in \mathbb{R}^m$ the test function

$$\varphi_x : \mathbb{R}^m \to \mathbb{C}, \qquad \varphi_x(y) = \varphi(x+y), \tag{29}$$

we see that

$$\int_{\mathbb{R}^m} (f * g)(x)\varphi(x)dx = \int_{\mathbb{R}^m} f(x)\left[\int_{\mathbb{R}^m} g(y)\varphi_x(y)dy\right] dx. \tag{30}$$

The formula (30) is in a form that we can extend to distributions. Indeed, the right-hand side of (30) suggests that we should define $u * v \in \mathcal{S}'(\mathbb{R}^m)$ as follows:

Definition 6. *If $u, v \in \mathcal{S}'(\mathbb{R}^m)$, then we define $u * v$ by*

$$\langle u * v, \varphi\rangle := \langle u, x \mapsto \langle v, \varphi_x\rangle\rangle. \tag{31}$$

To be more specific, we first map each $x \in \mathbb{R}^m$ to $\zeta(x) = \langle v, \varphi_x\rangle$, where φ_x is defined as in (29)*. We then apply u to this function ζ.*

This brings up a technical issue, as there is no guarantee that this ζ in Definition 6 is nice enough that we can feed it into u. Consequently, we require additional assumptions on u and v so that (31) makes sense in the first place.

The most standard assumption that one makes is that *either u or v has compact support.* If v is compactly supported, then (31) works as stated. On the other hand, if u is compactly supported, then we use the commutative property (25) as inspiration and define $u * v$ to be $v * u$, with the latter interpreted as in (31).

In some cases, one can assume less than compact support, but the exact conditions tend to be technical. However, since the distributions $\delta(\tau - |\xi|^2)$ fail to have compact support, this is in fact an essential point for Proposition 2.

3.2. **The Dispersive Convolution.** We now look at the left-hand side of (5):

$$\tilde{u} * \tilde{v} = \delta(\tau - |\xi|^2)\hat{f}(\xi) * \delta(\lambda - |\eta|^2)\hat{g}(\eta). \tag{32}$$

To apply Definition 6, we first fix $(\tau, \xi) \in \mathbb{R}^{1+n}$, and we look at

$$I(\tau, \xi) = \langle \delta(\lambda - |\eta|^2)\hat{g}(\eta), \varphi_{(\tau,\xi)}\rangle, \tag{33}$$

where, like in (29),

$$\varphi, \varphi_{(\tau,\xi)} \in \mathcal{S}(\mathbb{R}^{1+n}), \qquad \varphi_{(\tau,\xi)}(\lambda, \eta) = \varphi(\tau + \lambda, \xi + \eta).$$

However, from (17), we immediately obtain

$$I(\tau,\xi) = \int_{\mathbb{R}^n} \hat{g}(\eta)\varphi_{(\tau,\xi)}(|\eta|^2,\eta)d\eta = \int_{\mathbb{R}^n} \hat{g}(\eta)\varphi(\tau+|\eta|^2,\xi+\eta)d\eta. \tag{34}$$

Considering I now as a function, $I : \mathbb{R}^{1+n} \to \mathbb{C}$, we note that as long as $\hat{g}$ is in a nice enough space, for instance $\mathcal{S}(\mathbb{R}^n)$, then $I \in \mathcal{S}(\mathbb{R}^{1+n})$. As a result, we can apply $\tilde{u}$ now to I and hence make sense of Definition 6:

$$\langle \tilde{u} * \tilde{v}, \varphi\rangle = \langle \delta(\tau - |\xi|^2)\hat{f}(\xi), I\rangle = \int_{\mathbb{R}^n} \hat{f}(\xi)I(|\xi|^2,\xi)d\eta. \tag{35}$$

From (34) and (35), we see that

$$\langle \tilde{u} * \tilde{v}, \varphi\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\varphi(|\xi|^2+|\eta|^2,\xi+\eta)d\eta d\xi. \tag{36}$$

Applying a change of variables $\xi \mapsto \xi - \eta$ in the right-hand side of (36) yields

$$\langle \tilde{u} * \tilde{v}, \varphi\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi-\eta)\hat{g}(\eta)\varphi(|\xi-\eta|^2+|\eta|^2,\xi)d\eta d\xi. \tag{37}$$

3.3. **Completion of the Proof.** To complete the derivation of (5), we must work out its right-hand side. First, some clarification is in order as to how this quantity,

$$J = \int_{\mathbb{R}^n} \delta(\tau - |\xi-\eta|^2 - |\eta|^2)\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta,$$

is defined. To make sense of this, we first consider a fixed $\eta \in \mathbb{R}^n$, and we let

$$L_\eta = \delta(\tau - |\xi-\eta|^2 - |\eta|^2)\hat{f}(\xi-\eta)\hat{g}(\eta) \in \mathcal{S}'(\mathbb{R}^{1+n}). \tag{38}$$

(Here, η is fixed, while (τ,ξ) are the variables of the functions that this distribution acts on.) Then, we can naturally define J as the integral of the L_η's as follows:

$$\langle J, \varphi\rangle := \int_{\mathbb{R}^n} \langle L_\eta, \varphi\rangle d\eta. \tag{39}$$

It remains only to expand $\langle J, \varphi\rangle$. For this, we apply (17), (38), and (39):

$$\begin{aligned} \langle J, \varphi\rangle &= \int_{\mathbb{R}^n} \langle \delta(\tau - |\xi-\eta|^2 - |\eta|^2)\hat{f}(\xi-\eta)\hat{g}(\eta), \varphi\rangle d\eta \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi-\eta)\hat{g}(\eta)\varphi(|\xi-\eta|^2+|\eta|^2,\xi)d\xi d\eta. \end{aligned} \tag{40}$$

Comparing (37) with (40) yields (5) and completes the proof.

4. A Bilinear Estimate

As an application of Propositions 1 and 2, we prove a basic bilinear estimate for solutions of the free Schrödinger equation. The method of proof comes mainly from [2]; this estimate is also present as part of an exercise in [6].

Theorem 7. *Suppose u and v are solutions of* (1)*, with initial data given by* (4)*, and with f and g in sufficiently nice spaces. Furthermore, assume that:*

- *$\hat{f}$ is supported in the region $|\xi| \geq N$.*
- *$\hat{g}$ is supported in the region $|\xi| \leq M$.*

If $N \gg M$ (i.e., N is much larger than M), then the following estimate holds:

$$\|u \cdot v\|_{L_t^2 L_x^2(\mathbb{R}^{1+n})} \lesssim \frac{M^{\frac{n-1}{2}}}{N^{\frac{1}{2}}} \|f\|_{L_x^2} \|g\|_{L_x^2(\mathbb{R}^n)} \tag{41}$$

The iterated $L_t^q L_x^r$-norms are defined in the natural way:

$$\|u\|_{L_t^q L_x^r(\mathbb{R}^{n+1})} \lesssim_n \left[\int_{\mathbb{R}} \|u(t)\|_{L_x^r(\mathbb{R}^n)}^q dt \right]^{\frac{1}{q}}. \tag{42}$$

The obvious adjustments are made when $q = \infty$ or $r = \infty$.

This bilinear estimate provides a slight improvement over what one can derive using the usual Strichartz estimates for the Schrödinger equations. To see this, we first note that from Hölder's inequality and Sobolev embedding, we have

$$\|uv\|_{L_t^2 L_x^2} \lesssim_n \|u\|_{L_t^4 L_x^{\frac{2n}{n-1}}} \|v\|_{L_t^4 L_x^{2n}} \lesssim_n \|u\|_{L_t^4 L_x^{\frac{2n}{n-1}}} \||\nabla|^{\frac{n-2}{2}} v\|_{L_t^4 L_x^{\frac{2n}{n-1}}}, \tag{43}$$

where $|\nabla|$ is the square root of the negative spatial Laplacian on $\mathbb{R}^n$. (We suppress the domain $\mathbb{R}^{1+n}$ and later $\mathbb{R}^n$ for brevity.) Then, applying the standard Strichartz estimates (see [6]) and recalling the support restriction for $\hat{f}$, we obtain

$$\|uv\|_{L_t^2 L_x^2} \lesssim_n \|f\|_{L_x^2} \||\nabla|^{\frac{n-2}{2}} g\|_{L_x^2} \lesssim M^{\frac{n-2}{2}} \|f\|_{L_x^2} \|g\|_{L_x^2}. \tag{44}$$

Consequently, compared to (44), we gain an extra small factor of $(M/N)^{1/2}$ in (41), under the support assumptions for $\hat{f}$ and $\hat{g}$. In particular, the Strichartz estimate resulting in (44) cannot take advantage of the fact that $\hat{f}$ is nonzero only for high frequences. One can think of the improvement from the bilinear estimate as arising from examining more closely how u and v interact with each other in (spacetime) frequency space. More specifically, since $\hat{f}$ and $\hat{g}$ have vastly separated Fourier supports, the interactions between u and v in the product is limited. For certain problems, such as the well-posedness of nonlinear Schrödinger equations at low regularities, this slight gain is crucial; see, e.g., [2].

4.1. **The Dual Formulation.** We now focus our efforts on proving Theorem 7. We first reduce (41) to an estimate that we can attack directly.

By Plancherel's theorem, we see it suffices to prove that

$$\|\tilde{u} * \tilde{v}\|_{L_\tau^2 L_\xi^2} \lesssim \frac{M^{\frac{n-1}{2}}}{N^{\frac{1}{2}}} \|\hat{f}\|_{L_\xi^2} \|\hat{g}\|_{L_\xi^2} \tag{45}$$

(As usual, we use τ and ξ to denote the coordinates in frequency space.) However, in making sense of (45), we already encounter an issue: even though each of the spatial Fourier transforms $\hat{u}(t)$ and $\hat{v}(t)$, and hence their (spatial) convolutions $(\hat{u} *_\xi \hat{v})(t)$, are functions, there is no guarantee a priori that the *spacetime* Fourier transform $\tilde{u} * \tilde{v}$ must be a function, in particular one in L^2.

On the other hand, since $\tilde{u} * \tilde{v}$ is a distribution, we can feed it test functions. Suppose we can establish the following estimate for every $\varphi \in \mathcal{S}(\mathbb{R}^{1+n})$:

$$|\langle \tilde{u} * \tilde{v}, \varphi \rangle| \lesssim \frac{M^{\frac{n-1}{2}}}{N^{\frac{1}{2}}} \|\hat{f}\|_{L_\xi^2} \|\hat{g}\|_{L_\xi^2} \|\varphi\|_{L_\tau^2 L_\xi^2}. \tag{46}$$

Then, by the self-duality properties for L^2-spaces, and by the fact that the space of rapidly decreasing smooth functions is dense in L^2, it would follow that $\tilde{u} * \tilde{v}$ is represented by an L^2-function on $\mathbb{R}^{1+n}$, and also that its L^2-norm satisfies (45).

We now make use of the knowledge we gained from Propositions 1 and 2. More specifically, by recalling Proposition 2 and (37), we can expand (46) as

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi - \eta) \hat{g}(\eta) \varphi(|\xi - \eta|^2 + |\eta|^2, \xi) d\eta d\xi \right| \lesssim \frac{M^{\frac{n-1}{2}}}{N^{\frac{1}{2}}} \|\hat{f}\|_{L^2_\xi} \|\hat{g}\|_{L^2_\xi} \|\varphi\|_{L^2_\tau L^2_\xi}. \tag{47}$$

Therefore, to prove Theorem 7, it suffices to establish the above estimate (47).

4.2. **Further Reductions.** Let I denote the left-hand side of (47). Applying Hölder's inequality twice and recalling the supports of $\hat{f}$ and $\hat{g}$, we see that

$$\begin{aligned} I &\lesssim \|\hat{g}\|_{L^2_\xi} \left\{ \int_{|\eta| \leq M} \left[\int_{\mathbb{R}^d} \hat{f}(\xi - \eta) \varphi(|\xi - \eta|^2 + |\eta|^2, \xi) d\xi \right]^2 d\eta \right\}^{\frac{1}{2}} \\ &\lesssim \|\hat{f}\|_{L^2_\xi} \|\hat{g}\|_{L^2_\xi} \left[\int_{|\eta| \leq M} \int_{|\xi - \eta| \geq N} |\varphi(|\xi - \eta|^2 + |\eta|^2, \xi)|^2 d\xi d\eta \right]^{\frac{1}{2}}. \end{aligned}$$

In light of (47), it hence suffices to prove that

$$\int_{|\eta| \leq M} \int_{|\xi - \eta| \geq N} |\varphi(|\xi - \eta|^2 + |\eta|^2, \xi)|^2 d\xi d\eta \lesssim \frac{M^{n-1}}{N} \|\varphi\|^2_{L^2_\tau L^2_\xi}. \tag{48}$$

Given $1 \leq i \leq n$, we define the domain

$$D_i = \{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \mid |\eta| \leq M, \, |\xi^i - \eta^i| \geq d^{-1} N\}. \tag{49}$$

where ξ^i and η^i is the i-th components of ξ and η, respectively. Since

$$\bigcup_{i=1}^n D_i \supseteq \{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \mid |\eta| \leq M, \, |\xi - \eta| \geq N\}, \tag{50}$$

then to prove (47), and hence (41), we need only to show for each i that

$$\int_{D_i} |\varphi(|\xi - \eta|^2 + |\eta|^2, \xi)|^2 d\xi d\eta \lesssim \frac{M^{n-1}}{N} \|\varphi\|^2_{L^2_\tau L^2_\xi}. \tag{51}$$

For convenience, we will let J_i denote the left-hand side of (51).

4.3. **Completion of the Proof.** Consider the change of variables

$$(\xi, \eta) \mapsto (\tau = |\xi - \eta|^2 + |\eta|^2, \xi, \eta') \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n-1}, \tag{52}$$

where $\eta' \in \mathbb{R}^{n-1}$ denotes all the components of η except for η^i. Since the only actual change is $\eta^i \mapsto \tau$, it follows that the Jacobian for (52) is given by

$$\mathcal{J}_i = \left| \frac{\partial(\tau, \xi, \eta')}{\partial(\xi, \eta)} \right| = \left| \frac{\partial \tau}{\partial \eta^i} \right| = 2|(\xi^i - \eta^i) + \eta^i|. \tag{53}$$

Moreover, since we have on D_i that

$$|\eta^i| \leq M, \qquad |\xi^i - \eta^i| \geq d^{-1} N \gg M, \tag{54}$$

due to the definition (49), it follows that on D_i, we have

$$\mathcal{J}_i \simeq N. \tag{55}$$

Returning now to (51), we rewrite its left-hand side using (52):

$$J_i \leq \int_{|\eta'|\leq M} \left[\int_{\mathbb{R}} \int_{\mathbb{R}^n} \mathcal{J}_i^{-1} |\varphi(\tau,\xi)|^2 d\xi d\tau \right] d\eta'. \tag{56}$$

Finally, from (55), we see that

$$J_i \lesssim \int_{|\eta'|\leq M} d\eta' \cdot N^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}^n} |\varphi(\tau,\xi)|^2 d\tau d\xi \lesssim \frac{M^{n-1}}{N} \|\varphi\|^2_{L^2_\tau L^2_\xi}. \tag{57}$$

This completes the proof of (51) and hence (41).

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