On Controllability of Waves and Geometric Carleman Estimates

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Section 1

Background

Wave Equations

Main operators of interest:

- Wave operator (in $1 + n$ dimensions):

$$\Box := -\partial_t^2 + \Delta_x .$$

- With lower-order terms:

$$\mathcal{L} := \Box + \nabla_X + V, \qquad V(t,x) \in \mathbb{R}, \qquad X(t,x) \in \mathbb{R}^{1+n} .$$

Q. Why study wave equations?

- Prototypical example of hyperbolic PDEs.
- Many fundamental equations of physics contain wave behaviour.
  - Euler (fluids), Maxwell (electromagnetism), Einstein (gravity).

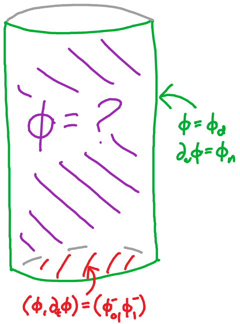
Solving Wave Equations

Wave equations on bounded domains:

- Spatial domain: $\Omega \subseteq \mathbb{R}^n$ (open, smooth boundary).
- Initial/final times: $T_- < T_+$.
- Wave equation: $\mathcal{L}\phi = F$ on $[T_-, T_+] \times \Omega$.

Well-posedness. Unique solution ϕ, given:

- Initial data: $(\phi, \partial_t\phi)|_{t=T_-}$.
- Boundary data: $\phi|_{(T_-,T_+)\times\partial\Omega}$ or $\partial_\nu\phi|_{(T_-,T_+)\times\partial\Omega}$.



A More Proactive Approach

Question (Control)

Can we control what happens to ϕ*?*

- *Given* *initial state* $(\phi, \partial_t \phi)|_{t=T_-} = (\phi_0^-, \phi_1^-)$...
- ...*achieve* *final state* $(\phi, \partial_t \phi)|_{t=T_+} = (\phi_0^+, \phi_1^+)$.

In real-world situations:

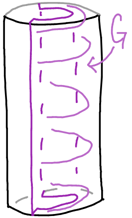
- We do not have unlimited powers.
- Can only directly affect part of system (the control).

Interior and Boundary Control

Boundary control: Steer ϕ through boundary data.

$$\mathcal{L}\phi = 0, \qquad \Gamma \subseteq (T_-, T_+) \times \partial\Omega.$$

- **Dirichlet**: $\phi|_{(T_-,T_+)\times\partial\Omega} = \chi_\Gamma \phi_d$.
- **Neumann**: $\partial_\nu \phi|_{(T_-,T_+)\times\partial\Omega} = \chi_\Gamma \phi_n$.



Interior control: Steer ϕ through forcing term.

$$\mathcal{L}\phi = \chi_\omega G, \qquad \omega \subseteq (T_-, T_+) \times \Omega.$$

Here, focus on Dirichlet boundary control.

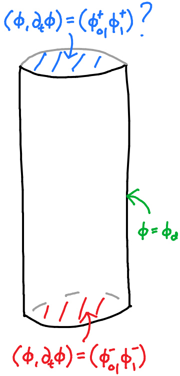
Exact Controllability

Problem (Exact Controllability)

Given: *Control region* $\Gamma \subseteq (T_-, T_+) \times \partial\Omega$

Goal: $\forall$ *initial/final data* $(\phi_0^\pm, \phi_1^\pm) \in L^2(\Omega) \times H^{-1}(\Omega)$*:*

- *Find* $\phi_d \in L^2(\Gamma)$*, such that solution of...*
    - *... wave equation:* $\mathcal{L}\phi|_{[T_-, T_+] \times \Omega} = 0$*...*
    - *... initial data:* $(\phi, \partial_t\phi)|_{t=T_-} = (\phi_0^-, \phi_1^-)$*...*
    - *... boundary data:* $\phi|_{(T_-, T_+) \times \partial\Omega} = \phi_d$*...*
- *... satisfies* $(\phi, \partial_t\phi)|_{t=T_+} = (\phi_0^+, \phi_1^+)$*.*



Q. Can solutions be controlled via Dirichlet boundary data?

Basic Observations

Observation. Can take $(\phi_0^+, \phi_1^+) = 0$ or $(\phi_0^-, \phi_1^-) = 0$.

- Null controllability.
- (From time-reversibility of wave equation.)

From now on, assume $(\phi_0^-, \phi_1^-) = 0$.

Observation. Finite speed of propagation.

- Information from $\partial\Omega$ needs time to travel to all of Ω.
- Fundamental lower bound for $T_+ - T_-$.

The Adjoint Problem

(P1) Consider linear operator

$$S : L^2(\Gamma) \to L^2(\Omega) \times H^{-1}(\Omega),$$
$$S(\phi_d) = (\phi, \partial_t \phi)|_{t=T_+}.$$

- **Goal:** Show S has full range.

$$\mathcal{L}\phi|_{(T_-,T_+)\times\Omega} = 0,$$
$$(\phi, \partial_t\phi)|_{t=T_-} = (0,0),$$
$$\phi|_{(T_-,T_+)\times\partial\Omega} = \phi_d.$$

(P2) Adjoint of S:

$$S^* : L^2(\Omega) \times H_0^1(\Omega) \to L^2(\Gamma),$$
$$S^*(\psi_1^+, \psi_0^+) = \partial_\nu \psi|_\Gamma.$$

$$\mathcal{L}^*\psi|_{(T_-,T_+)\times\Omega} = 0,$$
$$(\psi, \partial_t\psi)|_{t=T_+} = (\psi_0^+, \psi_1^+),$$
$$\psi|_{(T_-,T_+)\times\partial\Omega} = 0.$$

Observability and Controllability

By duality and closed range theorem:

- Controllable $\Leftrightarrow$ S surjective $\Leftrightarrow$ $\|\xi\| \lesssim \|S^* \xi\|$.
- Last statement known as observability inequality:

$$\|(\psi, \partial_t \psi)(T_+)\|_{H^1 \times L^2} \lesssim \|\partial_\nu \psi\|_{L^2(\Gamma)}, \qquad \psi|_{(T_-, T_+) \times \partial\Omega} = 0.$$

Main goal. Prove observability inequality.

(J.-L. Lions) Hilbert uniqueness method (HUM)

- Modern machinery for observability $\Rightarrow$ controllability.
- Methods for generating control ϕ_d (e.g. as minimizer of functional).
- Can find ϕ_d minimizing $L^2(\Gamma)$-norm.

Methods for Observability I

I. Fourier series methods

- Applies to one spatial dimension: $(-\partial_t^2 + \partial_x^2 + \alpha)\psi = 0$.
- Apply variants of Ingham's inequality on Fourier side.

Theorem (Ingham)

Consider sequence $(\lambda_n)_{n\in\mathbb{Z}}$ of real numbers, with

$$\lambda_{n+1} - \lambda_n \geq \gamma > 0, \qquad n \in \mathbb{Z}.$$

If $T > \frac{\pi}{\gamma}$, then for any sequence $(c_n)_{n\in\mathbb{Z}}$, we have

$$\sum_n |c_n|^2 \lesssim_{T,\gamma} \int_{-T}^{T} \left| \sum_n c_n e^{i\lambda_n t} \right|^2 dt.$$

Methods of Observability II

II. Multiplier methods

- Applies to all dimensions: $\Box\psi = 0$.
- **Idea.** Integrate by parts the RHS of

$$0 = \int_{[T_-, T_+]\times\Omega} \Box\psi \, \mathcal{S}_* \psi.$$

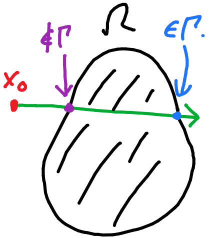
Theorem (Ho, Lions, ...)

Fix $x_0 \in \mathbb{R}^n$, and assume

$$T_+ - T_- > 2 \sup_{x\in\partial\Omega} |x - x_0|.$$

Then, for $\mathcal{L} := \Box$, observability holds with:

$$\Gamma := (T_-, T_+) \times \{x \in \partial\Omega \mid (x - x_0)\cdot \nu > 0\}.$$



Methods for Observability III

III. Carleman estimates

- Most robust technique.
- Extends multiplier methods to general $\mathcal{L}$.

Weighted (spacetime) integral estimates, with free real parameter.

$$\| e^{\lambda F} \nabla_{t,x} \psi \|_{L^2}^2 + \| e^{\lambda F} \psi \|_{L^2}^2 \lesssim \lambda^{-2} \| e^{\lambda F} \Box \psi \|_{L^2}^2 + \dots , \qquad \lambda \gg 1 .$$

- Take λ large $\Rightarrow$ absorb lower-order terms.

Theorem (Lasiecka–Triggiani–Zhang, Zhang, ...)

Previous theorem holds for general $\mathcal{L}$.

- *But, for technical reasons, also requires $x_0 \notin \bar{\Omega}$.*

Methods for Observability IV

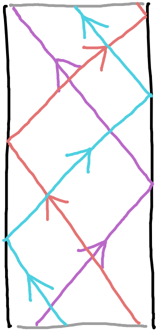
IV. Microlocal methods

- Most precise, optimal results (w.r.t. control region).
- Must also assume X, V analytic in t.

Theorem (Bardos–Lebeau–Rauch, Burq, ...)

Observability on Γ $\Leftrightarrow$ geometric control condition.

- *Each bicharacteristic (null geodesic) in $[T_-, T_+] \times \Omega$ hits Γ.*
- *Bicharacteristics reflect off boundary via geometric optics.*
- *(Le Rousseau–Lebeau–Terpolilli–Trélat) Time-dependent Γ.*



Section 2

Moving Boundaries

Time-Dependent Domains

Question (Main Question)

What about time-dependent domains?

$$\mathcal{U} = \bigcup_{T_- < \tau < T_+} (\{\tau\} \times \Omega_\tau).$$

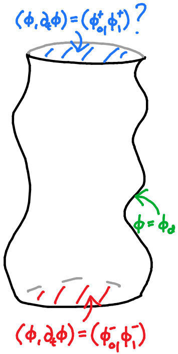
- *$\mathcal{U}$ has moving boundary:*

$$\mathcal{U}_b = \bigcup_{T_- < \tau < T_+} (\{\tau\} \times \partial\Omega_\tau).$$

- *Can we still prove control results?*

Assume boundary $\mathcal{U}_b$ is timelike:

- $\mathcal{U}_b$ "moves at less than wave/characteristic speed".
- $\mathcal{U}_b$ appropriate for boundary data.



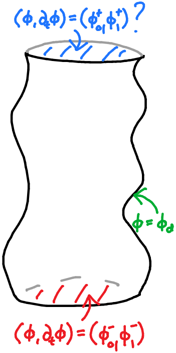
Controllability Revisited

Problem (Exact Dirichlet boundary controllability)

Given: *Control region* $\mathcal{Y} \subseteq \mathcal{U}_b$.

Goal: $\forall$ *initial/final data* $(\phi_0^\pm, \phi_1^\pm) \in L^2 \times H^{-1}$:

- *Find* $\phi_d \in L^2(\mathcal{Y})$ *that takes solution* ϕ...
- ... *from initial state* $(\phi, \partial_t \phi)|_{t=T_-} = (\phi_0^-, \phi_1^-)$...
- ... *to final state* $(\phi, \partial_t \phi)|_{t=T_+} = (\phi_0^+, \phi_1^+)$.



HUM. Exact controllability $\Rightarrow$ observability for adjoint system:

$$\| (\psi, \partial_t \psi)(T_+) \|_{H^1 \times L^2} \lesssim \| \partial_\nu \psi \|_{L^2(\mathcal{Y})}, \qquad \psi|_{\mathcal{U}_b} = 0.$$

Some Existing Results

Existing literature is sparse.

- Cannot use microlocal methods.

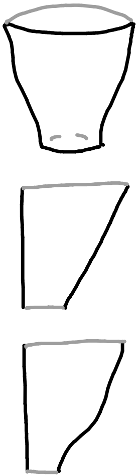
General n: (Only for $\mathcal{L} = \Box$).

- (Bardos–Chen) $\mathcal{U}$ expanding.
- (Miranda) $\mathcal{U}$ self-similar, asymptotically cylindrical.

$n = 1$: Recent works (only for $\mathcal{L} = \Box$).

- $\mathcal{U}_b$ = two lines (optimal), line + curve.
- (Cui–Jiang–Wang, Sun–Li–Lu, Wang–He–Li, ...)

Missing: General results in any dimension.



The Main Estimate (Rough Statement)

Theorem (S., 2019)

Consider general $\mathcal{L}$ on moving domain $\mathcal{U}$. Fix $x_0 \in \mathbb{R}^n$, and assume

$$T_+ - T_- > R_+ + R_- , \qquad R_\pm := \sup_{x \in \mathcal{U}_b \cap \{t = T_\pm\}} |x - x_0| .$$

Then, we have observability (for adjoint equation):

$$\| (\psi, \partial_t \psi) (T_\pm) \|_{H^1 \times L^2}^2 \lesssim \int_{\mathcal{Y}} |\partial_\nu \psi|^2 .$$

The observation region $\mathcal{Y}$ satisfies:

- *$\mathcal{Y}$ is a "much smaller", time-dependent set.*

Corollary

Exact controllability (for original equation), with control region $\mathcal{Y}$.

Some Remarks

First result for general timelike $\mathcal{U}_b$ and general $\mathcal{L}$.

- Improves existing Carleman/multiplier results for static $\mathcal{U}$.
  - $\mathcal{Y}$ "smaller than" $\{(x - x_0) \cdot \nu > 0\}$.
- x_0 can be outside or inside of domain.
- (Achieves GCC only when $n = 1$.)

Similar results for interior controllability:

- (Vaibhav Jena, 2020) Static domains, more regular controls.
- (Vaibhav Jena, in preparation) Time-dependent domains, less regular controls.

Plan for the Proof

I. Time-dependent domains.

- Consider model problem $\mathcal{L} = \Box$.
- Multiplier methods.

II. General operators $\mathcal{L}$.

- Proof of Carleman estimates.

III. Carleman $\Rightarrow$ observability.

- Exterior/interior observability.

Classical Multiplier Result

First, recall special case of free waves ($\mathcal{L} := \Box$).

- **Main step:** Integrate by parts, starting with

$$0 = \int_{[T_-, T_+] \times \Omega} \Box\psi \, \mathcal{S}^0_* \psi , \qquad \mathcal{S}^0_* \psi := (x - x_0) \cdot \nabla_x \psi + \frac{n-1}{2} \psi .$$

- Combine with energy conservation $\Rightarrow$

$$(T_+ - T_-) \mathcal{E}(T_\pm) \leq 2R \cdot \mathcal{E}(T_\pm) + \frac{1}{2} \int_{(T_-, T_+) \times \partial \Omega} [(x - x_0) \cdot \nu] |\partial_\nu \psi|^2 .$$

- $\mathcal{E}(t) = \frac{1}{2} \int_{\{t\} \times \Omega} |\nabla_{t,x} \psi|^2$.
- $R = \sup_{x \in \partial \Omega} |x - x_0|$.

- Green $\Rightarrow$ $T_+ - T_- > 2R$.
- Orange $\Rightarrow$ $\Gamma := \{(x - x_0) \cdot \nu > 0\}$.

Adaptation to Time-Dependent Setting

0. Main issue in time-dependent domains:

- Conservation of energy no longer holds.
- Integrations by parts, energy estimates $\Rightarrow$ additional boundary terms.

1. View in terms of spacetime (Minkowski) geometry.

- Minkowski spacetime:

$$(\mathbb{R}^{1+n}, g), \qquad g := -dt^2 + dx_1^2 + \dots dx_n^2.$$

- Replace reference point x_0 by reference event (t_0, x_0).

The Lorentzian Viewpoint

2. Replace classical multiplier $\mathcal{S}_*^0\psi$ by

$$\mathcal{S}_*\psi := [(x - x_0) \cdot \nabla_x\psi + (t - t_0)\, \partial_t\psi] + \frac{n-1}{2}\psi.$$

- ∂_t-part: corresponds to energy conservation arguments.

3. Apply Lorentzian version of integration by parts:

- $\mathcal{N}$: outward-pointing Minkowksi unit normal to $\mathcal{U}_b$.
- (Euclidean normal, with t-component reversed.)

The Multiplier Result

Theorem (S., 2019)

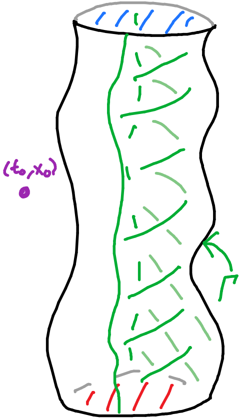
Consider $\mathcal{L} := \Box$ on moving domain $\mathcal{U}$. Assume

$$T_+ - T_- > R_+ + R_-, \qquad R_\pm := \sup_{x \in \mathcal{U}_b \cap \{t = T_\pm\}} |x - x_0|.$$

Then, we have observability (for adjoint problem):

$$\|(\psi, \partial_t \psi)(T_\pm)\|^2_{H^1 \times L^2} \lesssim \int_{\mathcal{Y}} |\mathcal{N}\psi|^2.$$

- $\mathcal{Y} = \mathcal{U}_b \cap \{\mathcal{N} f_0 > 0\}$.
- $\mathcal{N}$: *outward-pointing Minkowski unit normal on $\mathcal{U}_b$.*
- $f_0 := \frac{1}{4}[|x - x_0|^2 - (t - t_0)^2]$, *where...*
- *...t_0 satisfies $t_0 - T_- > R_-$ and $T_+ - t_0 > R_+$.*



Some Remarks

Proof is $\approx$ 2 pages.

$n = 1$: Recovers all existing results, and in full generality.

- Recovers optimal $T_+ - T_-$ (from GCC) for all $\mathcal{U}$.

General n: Handles general domains $\mathcal{U}$.

- Requires much smaller $T_+ - T_-$ than before (optimal in many cases).

$\{\mathcal{N} f_0 > 0\}$ generalises $\{(x - x_0) \cdot \nu > 0\}$:

$\mathcal{U}$ time-independent	$\mathcal{N} f_0 > 0 \Leftrightarrow (x - x_0) \cdot \nu > 0$
$\mathcal{U}$ "expanding" from t_0	Need smaller $\mathcal{Y}$
$\mathcal{U}$ "contracting" from t_0	Need larger $\mathcal{Y}$

Return to Carleman Estimates

Q. What about general $\mathcal{L}$?

Goal. Establish global Carleman estimate:

$$\| e^{\lambda F} (\nabla_{t,x} \psi, \psi) \|_{L^2}^2 \lesssim \lambda^{-2} \| e^{\lambda F} \Box \psi \|_{L^2}^2 + \| e^{\lambda F} \mathcal{N} \psi \|_{L^2 (\mathcal{Y})}^2 .$$

Integrate by parts the expression

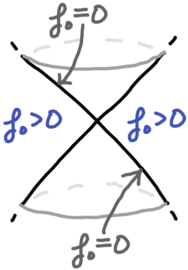
$$\int (e^{\lambda F} \Box \psi) \mathcal{S}_* (e^{\lambda F} \psi)$$

- Multiplier estimate for $e^{\lambda F} \Box e^{-\lambda F}$ and $\psi^\star := e^{\lambda F} \psi$.

The Carleman Weight

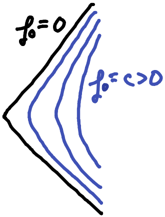
The weight $e^{\lambda F}$ is constructed from f_0: († Not quite true)

- $f_0 = \frac{1}{4}[|x - x_0|^2 - (t - t_0)^2]$.
- $e^{\lambda F} := f_0^{\lambda} \, e^{2\lambda b f_0^{1/2}}$.



Level sets of f_0 are hyperboloids:

- $f_0 = 0$: null cone about (t_0, x_0).
- $f_0 = c > 0$: one-sheeted hyperboloids.



$\mathcal{D}_0 := \{f_0 > 0\}$: null cone exterior.

A Novel Feature

New. $e^{\lambda F}$ vanishes at $f_0 = 0$.

- No boundary terms at $f_0 = 0$.
- Estimate supported on $\mathcal{D}_0 = \{f_0 > 0\}$.

This (roughly) yields the Carleman estimate:

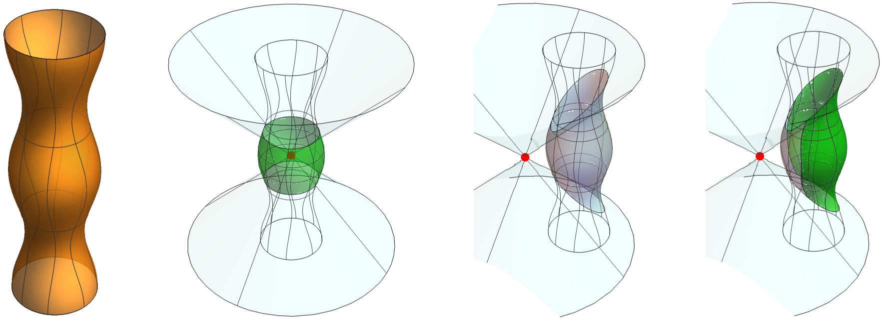
$$\|e^{\lambda F}(\nabla_{t,x}\psi, \psi)\|^2_{L^2(\mathcal{U}\cap\mathcal{D}_0)} \lesssim \lambda^{-2}\|e^{\lambda F}\Box\psi\|^2_{L^2(\mathcal{U}\cap\mathcal{D}_0)} + \int_{\mathcal{U}_b\cap\mathcal{D}_0} e^{2\lambda F}(\mathcal{N}f_0)|\mathcal{N}\psi|^2 .$$

- Energy inequalities $\Rightarrow$ observability, with

$$\mathcal{Y} := \mathcal{U}_b \cap \{\mathcal{N}f_0 > 0\} \cap \mathcal{D}_0 .$$

- (Classical Carleman methods: $\mathcal{Y} \approx \mathcal{U}_b \cap \{\mathcal{N}f_0 > 0\}$.)
- Extra restriction of $\mathcal{Y}$ to $\mathcal{D}_0$.

Some Pictures



- Left: $\mathcal{U}_b$.
- Red dot: (t_0, x_0).
- Purple: $\mathcal{U}_b \cap \mathcal{D}_0$.
- Green: $\mathcal{Y} := \mathcal{U}_b \cap \{\mathcal{N} f_0 > 0\} \cap \mathcal{D}_0$.

A Pseudoconvexity Issue

Main requirement for Carleman estimates is pseudoconvexity.

- Level sets of f_0 (barely) fail to be pseudoconvex.

Thus, can only obtain a degenerate Carleman estimate:

$$\| e^{\lambda F} \psi \|^2_{L^2 (\mathcal{U} \cap \mathcal{D}_0)} \lesssim \lambda^{-2} \| e^{\lambda F} \Box \psi \|^2_{L^2 (\mathcal{U} \cap \mathcal{D}_0)} + \int_{\mathcal{U}_b \cap \mathcal{D}_0} e^{2 \lambda F} (\mathcal{N} f_0) | \mathcal{N} \psi |^2 .$$

- No H^1-control for $\psi \Rightarrow$ no observability.
- Thus, cannot build weight $e^{\lambda F}$ from f_0.

Idea. Perturb $f_0 \rightarrow f_\varepsilon$ such that...

- ... level sets of f_ε are pseudoconvex.

The Classical Approach

Classical Carleman observability results:

$$f_\varepsilon := \frac{1}{4}[|x - x_0|^2 - (1 - \varepsilon)(t - t_0)^2].$$

- Hyperboloids for waves with slower speed.

Drawback. Not well-adapted to characteristics of wave equation.

- $f_\varepsilon = 0$ no longer the null cone about (t_0, x_0).
- Cannot obtain Carleman estimates supported on $\mathcal{D}_0$.

Drawback. Only works for $(t_0, x_0) \notin \mathcal{U}$.

- (Arises from dealing with the region $\{f_0 < 0\}$.)

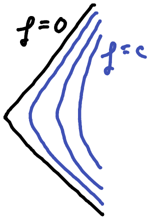
A Conformal Viewpoint

New idea. Perturb geometry rather than f_0:

- Consider "warped" Minkowski metric:

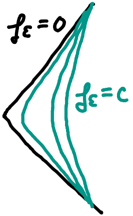
$$g_\varepsilon := -dt^2 + dr^2 + (r + 2\varepsilon f_0)^2 \mathring{\gamma}, \qquad (t_0, x_0) = 0, \quad \varepsilon > 0.$$

- Level sets of f_0 pseudoconvex w.r.t. g_ε-geometry.



Observation. $(\mathcal{D}_0, g)$, $(\mathcal{D}_0, g_\varepsilon)$ conformally related.

- Pseudoconvexity is conformally invariant.
- $f_\varepsilon :=$ pullback of f_0 through conformal isometry.



Strategy. 2 steps for proof of Carleman estimate:

1. Prove "warped" Carleman estimate on $(\mathcal{D}_0, g_\varepsilon)$.
2. Pull "warped" Carleman estimate back to $(\mathcal{D}_0, g)$.

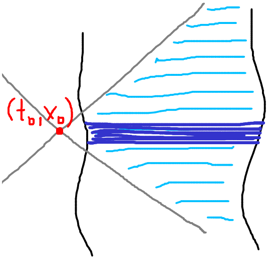
Carleman to Observability

Idea. Carleman + energy estimate $\Rightarrow$ observability.

- But, Carleman weight $e^{\lambda F}$ vanishes on boundary of $\mathcal{D}_0$.
- Cannot capture energy where $e^{\lambda F}$ vanishes.

Exterior observability. $(t_0, x_0) \notin \mathcal{U}$.

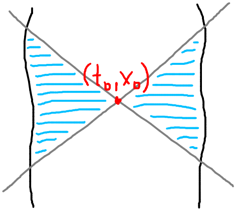
- Cross-sections of $\mathcal{U}$ near $t = t_0$ satisfy $e^{\lambda F} \gtrsim 1$.
- $\Rightarrow$ Carleman controls energy near $t = t_0$.



Interior Observability

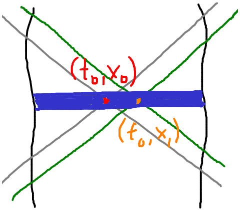
Interior observability. $(t_0, x_0) \in \mathcal{U}$.

- $e^{\lambda F}$ vanishes at every time in $\mathcal{U}$.
- Carleman does not control energy at any time.



Idea. Use *two* Carleman estimates:

- Centred at nearby points (t_0, x_0), (t_0, x_1).
- *Sum* of estimates controls energy near $t = t_0$.



The Main Observability Estimate

Theorem (S., 2019)

Consider general $\mathcal{L}$ on moving domain $\mathcal{U}$. Assume

$$T_+ - T_- > R_+ + R_-, \qquad R_\pm := \sup_{x \in \mathcal{U}_b \cap \{t = T_\pm\}} |x - x_0|.$$

Then, we have observability (for adjoint problem):

$$\| (\psi, \partial_t \psi) (T_\pm) \|_{H^1 \times L^2}^2 \lesssim \int_{\mathcal{Y}} |\mathcal{N} \psi|^2 .$$

- $\mathcal{Y}$*: any open subset of boundary with*

$$\mathcal{Y} \supseteq \overline{\mathcal{U}_b \cap \{ \mathcal{N} f_0 > 0 \} \cap \mathcal{D}_0} .$$

- f_0 *and* t_0 *as before* ($t_0 - T_- > R_-$ *and* $T_+ - t_0 > R_+$).

Remark. Any such $\mathcal{Y}$ suffices:

- Perturbation $f_0 \to f_\varepsilon$ can be arbitrarily small.

Conclusions

1. Controllability for waves on time-dependent domains.

- First general result.
- Applies to general $\mathcal{U}$ and (non-analytic) $\mathcal{L}$.

2. Best known results using Carleman methods.

- Smaller control region (restriction to null cone exterior).
- Leads to improved results for time-independent domains.

3. Proof uses geometric ideas and intuitions.

- Extends to geometric wave equations.

Hyperbolic PDEs

What about other (2nd order) hyperbolic PDE?

$$-\partial_t^2 + \Delta_x \qquad \Rightarrow \qquad -\partial_t^2 + \sum_{1 \leq i,j \leq n} a^{ij} \partial_i \partial_j .$$

- Or, what about geometric wave equations on manifolds?

Classical methods already extend to time-independent settings:

$$\mathcal{U} := \mathbb{R} \times \Omega, \qquad a^{ij} = a^{ij}(x).$$

- Many results using multiplier, Carleman, and microlocal methods.

Outlook and Upcoming Work

Question

What about time-dependent settings: $a^{ij} = a^{ij}(t, x)$?

- *(In other words, geometric waves on Lorentzian manifolds.)*
- *Open question, not explored in existing literature.*

Idea. Geometric tools can be used to treat this problem.

- Use methods described for previous theorem...
- ...but on Lorentzian geometry defined by the PDE.
- (Joint work in progress with Vaibhav Jena.)