A Brief Introduction to Mathematical Relativity

Arick Shao

Imperial College London

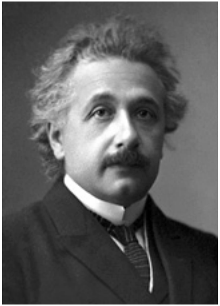
Einstein's Postulates

(A. Einstein, 1905) Postulates of special relativity:*

1. Relativity principle: *The laws of physics are the same in all inertial frames of reference.*
2. Speed of light: *The speed of light in vacuum has the same value c in all inertial frames of reference.*

Postulates + physical considerations ⇒:

- *Observers moving at different velocities will perceive length, time, etc., differently.*



A. Einstein (1879–1955)⋆

* Quoted from Nobelprize.org.

⋆ Photo from Nobelprize.org.

Minkowski's Formulation

(1907) Hermann Minkowski:

- Geometric formulation of special relativity.
- Ideas later extended to general relativity.

Time ($\mathbb{R}$) + space ($\mathbb{R}^3$) = *spacetime* ($\mathbb{R}^4$)

- More accurately, $\mathbb{R}^4$ "modulo coordinate systems."
- Formally, $\mathbb{R}^4$ *as a (differential) manifold*.
- (Newtonian theory: $\mathbb{R} \times \mathbb{R}^3$)

H. Minkowski (1864–1909)*

* Photo from `www.spacetimesociety.org`.

Euclidean vs. Minkowski

4-d Euclidean space $(\mathbb{R}^4, \delta)$:

- Euclidean (square) distance:

$$d^2(p,q) := \sum_{k=1}^{4} (p^k - q^k)^2.$$

- Corresponding differential structure (*Euclidean metric*):

$$\delta := dx^2 + dy^2 + dz^2 + dw^2.$$

- For vectors $u, v \in \mathbb{R}^4$:

$$\delta(u,v) = \sum_{k=1}^{4} u^k v^k.$$

- Riemannian manifold

4-d Minkowski spacetime $(\mathbb{R}^4, \eta)$:

- Minkowski (square) "distance":

$$d^2(p,q) := -(q^0 - p^0)^2 + \sum_{k=1}^{3} (q^k - p^k)^2.$$

- Corresponding differential structure (*Minkowski metric*):

$$\eta := -dt^2 + dx^2 + dy^2 + dz^2.$$

- For vectors $u, v \in \mathbb{R}^4$:

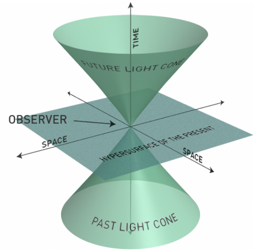
$$\eta(u,v) = -u^0 v^0 + \sum_{k=1}^{3} u^k v^k.$$

- Lorentzian manifold

Causal Character

Geometry of $(\mathbb{R}^4, \eta)$ radically different from that of $(\mathbb{R}^4, \delta)$.

- Lack of sign definiteness $\Rightarrow$ different directions have different meanings.



Causal character: A vector $v \in \mathbb{R}^4$ is

- *Spacelike* if $\eta(v, v) > 0$ or $v = 0$.
- *Timelike* if $\eta(v, v) < 0$.
- *Null* (*lightlike*) if $\eta(v, v) = 0$ and $v \neq 0$.

Physical interpretations:

- *Observer*: timelike curve.
- *Light*: null lines.

* Image by Stib on en.wikipedia.org.

Relativity

Many concepts have no absolute prescription:

- Elapsed time, length, energy-momentum.
- Only makes sense relative to an observer.

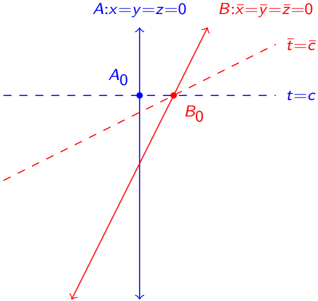
Observer $\bar{O} \Rightarrow$ coordinates $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$ adapted to $\bar{O}$.

- $\bar{x} = \bar{y} = \bar{z} = 0$ along $\bar{O}$.
- Observer can measure *with respect to these coordinates*.
- Constant velocity $\Rightarrow$ *inertial coordinate system*:

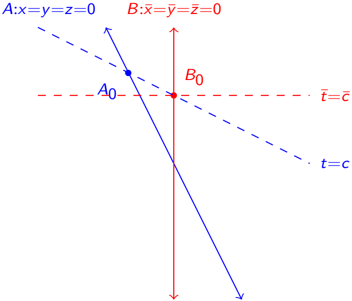
$$\eta = -d\bar{t}^2 + d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2 .$$

Simultaneity

Observers moving at different (constant) velocities will perceive different events to be "at the same time."



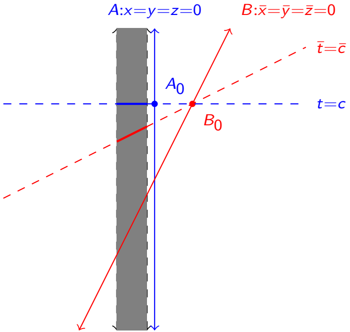
Coordinates with observer A at rest.



Coordinates with observer B at rest.

Length Contraction

Observers moving at different velocities perceive lengths differently.



Observers A and B measure a rod (at rest with respect to A).

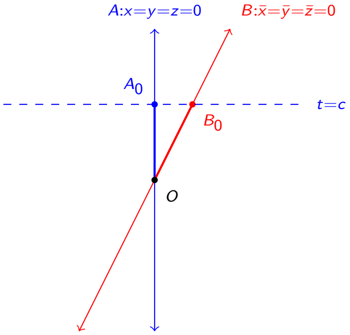
Shaded region represents rod.

- A measures "length" of blue bolded segment through rod.
- B measures "length" of red bolded segment through rod.

B measures shorter length than A.

Time Dilation

Clocks moving at different velocities observed to tick at different speeds.



Observer A measures clocks carried by both A and B.

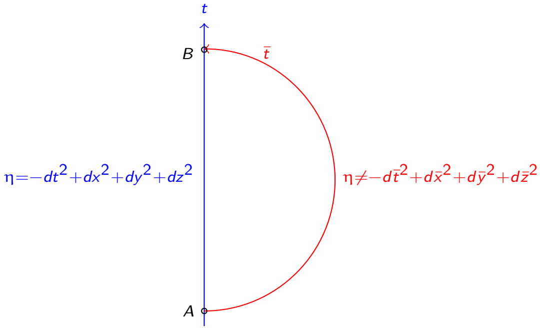
Both A and B carry clock.

- Both clocks synchronised at O.
- A measures both clocks at $t = c$.

A measures less time elapsed on B's clock than A's clock.

Twin Paradox

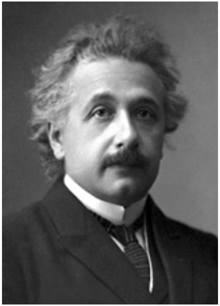
Different timelike curves between two events will have different lengths.



From A to B: more time elapses for t-observer than for $\bar{t}$-observer.

Geometry and Gravity

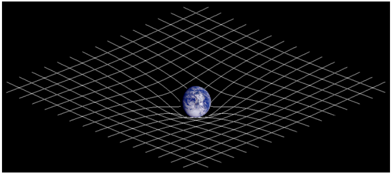
Special relativity does not model gravity.



A. Einstein (1879–1955)

(A. Einstein, 1915) *General relativity*:

- Gravity not modeled as a force, but rather through geometry of spacetime.
- Revolutionary idea: *gravity* $\Leftrightarrow$ *curvature*



Curved spacetime, with gravity represented by spacetime curvature.*

* Image by Johnstone on en.wikipedia.org.

Spacetimes

Extend notion of spacetime:

- $(\mathbb{R}^4, \eta) \mapsto$ 4-dimensional *Lorentzian manifold* (M, g).
- Geometric content: *Lorentzian metric* g on M.
- g has "same signature $(-1, 1, 1, 1)$" as η.*

Study of spacetimes $\Leftrightarrow$ *Lorentzian geometry*:

- Analogue of Riemannian geometry.
- Lines in $\mathbb{R}^4 \mapsto$ geodesics
- Can formally make sense of *curvature*.

* At each $p \in M$, we have a bilinear form $g|_p$ on $T_p M$ of signature $(-1, 1, 1, 1)$.

Physical Interpretations

Principle of covariance: *physical laws are intrinsic properties of the manifold* (M, g)*, i.e., independent of coordinates on* M*.*

Causal character for tangent vectors:

- v is spacelike if $g(v, v) > 0$ or $v = 0$.
- v is timelike if $g(v, v) < 0$.
- v is null if $g(v, v) = 0$ and $v \neq 0$.

Observers: timelike curves.

- Free fall: timelike geodesics.
- Light: null geodesics.

Matter Fields

Gravity closely coupled to matter via the *Einstein field equations*:

$$\mathrm{Ric}_g - \frac{1}{2} \mathrm{Sc}_g \, g = T .$$

- Ric_g: *Ricci curvature* associated with g.
- Sc_g: *Scalar curvature* associated with g.
- T: *Stress-energy tensor* associated with matter field Φ.
- Φ: satisfies equations according to its physical theory.

No matter field $\Rightarrow$ *Einstein-vacuum equations* (*EVE*):

$$\mathrm{Ric}_g = 0.$$

Connection to Differential Equations

Question

How do we interpret the EVE?

Write equations in terms of g and a fixed coordinate system on M:

- 2nd-order quasilinear system of PDE for components of g:

$$0 = -\frac{1}{2} \sum_{\alpha,\beta} g^{\alpha\beta} (\partial_\alpha \partial_\beta g_{\mu\nu} - \partial_\beta \partial_\nu g_{\mu\alpha} - \partial_\beta \partial_\mu g_{\nu\alpha} + \partial_\mu \partial_\nu g_{\alpha\beta}) + \frac{1}{2} \sum_{\alpha,\beta,\gamma,\delta} g_{\mu\nu} g^{\alpha\beta} g^{\gamma\delta} (\partial_\alpha \partial_\beta g_{\gamma\delta} - \partial_\beta \partial_\gamma g_{\alpha\delta}) + F_0(g, \partial g). \tag{1}$$

Q. What is the character of (1)? (elliptic, parabolic, hyperbolic)

- Determines what types of problems are reasonable to solve.

Special Coordinates

Bad news: In general, (1) is none of the above.

In special coordinates, (1) becomes hyperbolic.

$$0 = -\frac{1}{2} \sum_{\alpha,\beta} g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu} + F_1(g, \partial g). \qquad (2)$$

- Should be solved as an "initial value problem".
- (1952, Y. Choquet-Bruhat) Solved Einstein-vacuum equations for short times.

Y. Choquet-Bruhat (b. 1923)*

*Photo by Renate Schmid for the Oberwolfach Photo Collection (`owpdb.mfo.de`).

Well-Posedness

Question

Is the initial value problem **well-posed**?

Given initial data, can we:

1. *Show existence of solution to EVE?*
2. *Show uniqueness of this solution?*
3. *Show continuous dependence of solution on initial data?*

In other words, given the state of the universe at some time, can we:

- (1) + (2): Predict the future/past?
- (3): Approximately predict the future/past?

Solving the Equations

Many difficulties behind solving the EVE:

- Equations are highly nonlinear.
- Initial data must first satisfy (elliptic) constraint equations.

Note: Unlike other PDE, *we are solving for the spacetime itself*!

- Usually, solve for functions on fixed background (e.g., $\mathbb{R}^N$).
- Here, we solve for (M, g), i.e., the "universe".

Example

Initial data: Euclidean space $(\mathbb{R}^3, \delta)$
Solution: Minkowski spacetime $(\mathbb{R}^4, \eta)$

Gravitational Waves

The EVE, in the form (1), are hyperbolic (i.e., "like wave equations").

- (Also, linearisation of EVE about Minkowski spacetime yields wave equations.)

Thus, expect wave-like behaviour for spacetimes (radiation, etc.):

- Early prediction of *gravitational waves*.
- Recently observed by LIGO.

Cosmological Constant

Can add extra term to Einstein equations

$$\operatorname{Ric}_g - \frac{1}{2} \operatorname{Sc}_g g - \Lambda g = T .$$

- $\Lambda \in \mathbb{R}$: *cosmological constant*.

Taking $\Lambda \neq 0$ produces solutions with very different properties.

- $\Lambda > 0$: *De Sitter*
- $\Lambda < 0$: *Anti-de Sitter (AdS)*

Cosmology

Assume spacetime is *homogeneous* and *isotropic*:

$$g = -dt^2 + a(t) \cdot d\Sigma.$$

- Independent of space and direction.
- Approximates universe at large scales.

Consider Einstein equations, coupled to "dust" matter:

- Equations becomes ODE in time.
- Given initial data, solve *backwards* in time $\Rightarrow$ *Friedmann–Lemaître–Robertson–Walker* (*FLRW*) *spacetime* (1920s, 1930s).

After *finite* time, universe "shrinks to nothing" (i.e., $a(t) \to 0$).

- Early model of *big bang* singularity.

Schwarzschild Spacetimes

(1916) *Schwarzschild spacetimes*: spherically symmetric solution of EVE:

$$g = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2).$$

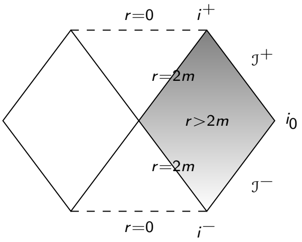
K. Schwarzschild (1873–1916)*

First interpretation: region outside a spherical object with mass m.

- Observe: equation for g dies at $r = 2m$ and $r = 0$.

*Photo from en.wikipedia.org.

Global Schwarzschild



Maximal Schwarzschild spacetime (modulo spherical symmetry).

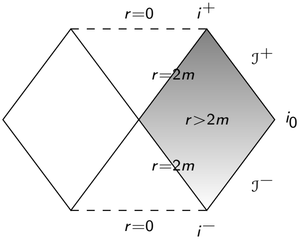
Schwarzschild can be interpreted purely as vacuum spacetime:

- $r = 2m$ is not a real singularity (*coordinates* fail, but *manifold* can be extended through $r = 2m$).
- $r = 0$ is a real singularity (scalar curvature dies).

Maximal extension looks like figure:

- Two copies of outer region $r > 2m$.

Schwarzschild Black Holes



Maximal Schwarzschild spacetime (modulo spherical symmetry).

$r = 2m$ called the *event horizon*:

- No observer or light ray entering $r < 2m$ can leave.
- Any timelike or null geodesic starting in $r < 2m$ terminates at $r = 0$ in finite (proper) time.

First models of *black holes* and *singularities*.

More general family of vacuum spacetimes:

- (1963) *Kerr spacetimes*: rotating black holes.

Singularity Theorems

Question

Are singularities an artifact of very special spacetimes?

(1965) *Penrose singularity theorem*:

- *Trapped surfaces* + other generic conditions $\Rightarrow$ singularity
- Trapped surface: all emanating light rays are pulled closer together.
- Example: Schwarzschild, spheres within $r < 2m$.

Moreover, singularity formation can be dynamic:

- (2009) D. Christodoulou: constructed "nice" initial data, from which a trapped surface eventually forms.

Cosmic Censorship Conjectures

Question (Open)

What is the nature of general singularities?
How do they form?

(1969, R. Penrose) Conjecture: if a singularity forms, it should be hidden within a black hole.

Problem (Cosmic Censorship (CC))

- *Weak cosmic censorship (WCC): singularities hidden within event horizon, hence unseen from outside.*
- *Strong cosmic censorship (SCC): general relativity is deterministic (we can predict the future).*

R. Penrose (b. 1931)*

*Photo by Festival della Scienza on `en.wikipedia.org`.

Cosmic Censorship, Revised

CC encounters some major obstacles:

1. Neither WCC nor SCC implies the other.
2. Both WCC and SCC are false.

Problem

Under reasonable generic conditions (to be determined), ...

In general, these are open problems.

- The correct mathematical formulation is unclear.

Asymptotic States

Question (Open)

Can we describe the long-time dynamics and asymptotics of vacuum spacetimes, i.e., solutions of EVE?

- *What is "the end state of the universe"?*

Problem (Final State Conjecture)

For general initial data that is "asymptotically flat", the solution spacetime should asymptotically settle down to:

- *Minkowski spacetime, or*
- *One or more Kerr black holes solutions, moving apart from each other.*

Very difficult problem $\Rightarrow$ first consider special cases.

Minkowski Spacetime

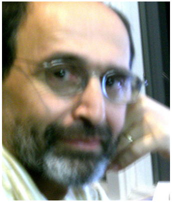
First step: *global stability* of Minkowski spacetime.

- Study solutions close to Minkowski spacetime.

(1993) D. Christodoulou, S. Klainerman:

- If initial data is "close to Euclidean space" $\mathbb{R}^3$, then the solution of EVE is "close to Minkowski spacetime."
- Solution spacetime "decays to Minkowski" at infinity.
- (Theorem and proof: 526-page book.*)

D. Christodoulou (b. 1951)⋆



S. Klainerman (b. 1950)†

*D. Christodoulou and S. Klainerman, *The Global Nonlinear Stability of the Minkowski Space*, Princeton University Press, 1994

⋆Photo from ETH Zurich www.math.ethz.ch.

†Photo from personal website web.math.princeton.edu.

Black Hole Spacetimes

Problem (Stability of Kerr Spacetimes)

Are Schwarzschild and Kerr spacetimes similarly stable?

This is a major open problem.

- Significant progress in recent years.

(2016) M. Dafermos, G. Holzegel[†], I. Rodnianski

- Stability of Schwarschild spacetime for a *linearisation* of EVE about Schwarzschild.
- Linearised solutions decay to (linearised) Kerr spacetime.

[†] Imperial College London

The End

Thank you for your attention!