A Generalized Representation Formula for Tensor Wave Equations on Curved Spacetimes

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March 8, 2012

The Model Equation

- Consider first the Minkowski spacetime $\mathbb{R}^{1+3}$.
- Consider the (scalar) wave equation,

$$\Box\phi = -\partial_t^2\phi + \Delta\phi = \psi, \qquad \phi, \psi \in C^\infty(\mathbb{R}^{1+3}),$$

with initial data

$$\phi|_{t=0} = \alpha_0 \in C^\infty(\mathbb{R}^3), \qquad \partial_t\phi|_{t=0} = \alpha_1 \in C^\infty(\mathbb{R}^3).$$

- One has an explicit solution for ϕ – *Kirchhoff's formula* – in terms of ψ, α_0, and α_1.

The Model Formula

- Write $\phi = \phi_1 + \phi_2$, where
  - ϕ_1 satisfies $\Box\phi = \psi$, with zero initial data.
  - ϕ_2 satisfies $\Box\phi \equiv 0$, with initial data α_0, α_1.
- Then, we have the representation formula

$$\phi_2(t, x) = \frac{1}{4\pi t^2} \int_{\partial B(x,t)} [\alpha_0(y) + (y - x) \cdot \nabla\alpha_0(y)] d\sigma_y + \frac{1}{4\pi t} \int_{\partial B(x,t)} \alpha_1(y) d\sigma_y,$$

$$\phi_1(t, x) = \frac{1}{4\pi} \int_0^t \int_{\partial B(x,r)} \frac{\psi(y, t - r)}{r} d\sigma_y dr.$$

  - $B(x, r)$ is the ball in $\mathbb{R}^3$ about x of radius r.

Curved Spacetimes

- **Main Question:** Can we extend this representation to geometric settings, i.e., to curved spacetimes?
  - Curved spacetime: any general $(1+3)$-dimensional Lorentzian manifold (M, g).
  - Equation: *Covariant tensorial* wave equation,

$$\Box_g \Phi = g^{\alpha\beta} D^2_{\alpha\beta} \Phi = \Psi,$$

    with appropriate "initial conditions".
  - Goal: representation formula

$$\Phi|_p = F(\Psi) + \text{error}(\Phi) + \text{initial data.}$$

- Some classical applications:
  1. (Y. Choquét-Bruhat) Local well-posedness of the Einstein-vacuum equations.
  2. (Chruściel-Shatah) Global existence of the Yang-Mills equations in curved spacetimes.

Infinite-Order Formulas

- Infinite-order, or "Hadamard-type", representation formulas are more explicit and precise.
  - Require infinitely many derivatives of metric g.
  - Formula is only local: *require geodesic convexity*.
- Wave equations in curved spacetimes no longer satisfy the *strong Huygens principle*.
  - Representation formula at point p depends on entire causal, rather than null, past (or future) of p.
- These severe restrictions for infinite-order formulas often make them undesirable for nonlinear PDEs.

First-Order Formulas

- In contrast, one can also derive first-order, or "Kirchhoff-Sobolev-type", representation formulas.
  - Again, formula is only local.
  - Require only limited number of derivatives of g.
  - Formula not explicit – contains recursive error terms:

$$\Phi|_p = F(\Psi) + \text{error}(\Phi) + \text{initial data}.$$

- Representation formula can be supported on only the null past (or future) of p.
  - Require smoothness of null, rather than causal, cone.
  - The price to be paid is the recursive error terms.

A Recent Result

- "Kirchhoff-Sobolev Parametrix" [KSP] (Klainerman-Rodnianski, 2007): first-order representation formula on curved spacetimes.
  - Valid within *null radius of injectivity*.
  - Supported entirely on past null cone.
  - Handles covariant tensorial wave equations, using only fully covariant (coordinate-independent) techniques.
  - Extendible to wave equations on vector bundles.
- Rough statement of KSP:

$$4\pi \cdot g(\Phi|_p, J_p) = \int_{\mathcal{N}^-(p)} [g(A, \Psi) + Err(A, \Phi)] + i.v..$$

  - A corresponds to r^{-1} in Minkowski space.

Applications of KSP

- Applications of this formula:
  1. Gauge-invariant proof of global existence of Yang-Mills.
     - The classical result (Eardley-Moncrief, 1982) relies on Cronström gauge.
  2. Breakdown criterion for Einstein-vacuum equations (Klainerman-Rodnianski, 2010).
     - Needed pointwise bound for Riemann curvature R of (M, g), which satisfies tensor wave equation

$$\Box_g R = R \cdot R.$$

Extending KSP

- The main result of this presentation is a generalization of KSP, which we call [GKSP].
- **Q.** Why generalize KSP?
  1. Want to handle *systems* of tensor wave equations with (covariant) *first-order terms*:

$$\Box_g \Phi_{\mathbf{m}} + \sum_{\mathbf{c}=1}^{\mathbf{n}} P_{\mathbf{m}}{}^{\mathbf{c}} \cdot D\Phi_{\mathbf{c}} = \Psi_{\mathbf{m}}, \qquad 1 \leq \mathbf{m} \leq \mathbf{n}.$$

  2. Removal of extraneous assumptions needed in KSP.
  3. Explicit formula for initial value terms.

Handling First-Order Terms

- Analogous breakdown criterion for Einstein-Maxwell equations (S., 2010)
  - Curvature R and electromagnetic tensor F satisfy

$$\Box_g R \cong F \cdot D^2 F + (R + DF)^2 + l.o.,$$
$$\Box_g DF \cong F \cdot DR + (R + DF)^2 + l.o..$$

  - Right hand side has first-order terms.
    - In KSP, these become part of the inhomogeneity Ψ, but this does not yield the necessary estimates.
    - For GKSP, we must treat these terms differently.

Removing Assumptions

- Assumptions for KSP:
  1. Smoothness/regularity of all past null cones in a neighborhood of the base point p.
  2. Local hyperbolicity – spacelike “initial” hypersurface passed by null cone exactly once.
- Less assumptions for GKSP:
  1. Smoothness/regularity of past null cone from p.

  - (1) for KSP weakened to only null regularity at p.
  - (2) for KSP is not needed at all.

A Different Proof

- Why can we weaken these assumptions?
- We use a different proof for GKSP.
  - Proof of KSP uses an *optical function* u, whose level sets form a foliation of null cones.
  - Proof of KSP uses distributions: derivatives of δ composed with u.
  - Proof of GKSP remains entirely on the null cone from p.
  - Proof of GKSP avoids distribution theory, uses more rigorous calculus and selective integrations by parts.

Remaining on the Null Cone

- In both KSP and GKSP, except for Φ, Ψ, and the first order coefficients P (in GKSP), *all other quantities are defined only on the past null cone from p*.
  - Both KSP and GKSP supported on the null cone.
- In proving KSP:
  - Integration by parts for all derivatives – results in terms not defined only on null cone.
  - These terms disappear due to miraculous cancellations.
- In proving GKSP:
  - Integration by parts only for derivatives tangent to null cone – thus, we never see terms off the null cone.
  - The null support property is a natural consequence, rather than a miracle.

GKSP, Preliminary Version

- GKSP given roughly as follows:

$$4\pi \cdot \sum_{\mathbf{m}=1}^{\mathbf{n}} g(\Phi_{\mathbf{m}}|_p, J_p^{\mathbf{m}})$$
$$= \int_{\mathcal{N}^-(p)} \left[\sum_{\mathbf{m}=1}^{\mathbf{n}} g(A^{\mathbf{m}}, \Psi_{\mathbf{m}}) + Err(A, \Phi, P) \right] + i.v..$$

  - $J_p^{\mathbf{m}}$: tensor field at p.
  - $A^{\mathbf{m}}$: satisfies tensorial transport equation (depending on P and the geometry of $\mathcal{N}^-(p)$) along null generators of $\mathcal{N}^-(p)$, with initial value determined by $J_p^{\mathbf{m}}$.
  - i.v.: initial value terms – integrals involving A and Φ (and first derivatives), on "lower boundary" of $\mathcal{N}^-(p)$.

Vector Bundle Extensions

- Both KSP and GKSP can be directly generalized to vector bundles over M, with a bundle metric and a compatible bundle connection.
  - Application: Handling Yang-Mills and Einstein-Yang-Mills equations.

Foliation of the Null Cone

- Fix a *foliating function* f on null cone $\mathcal{N}^-(p)$:
  - $f > 0$, with $f \to 0$ at p.
  - f increasing along past null generators.
  - f foliates $\mathcal{N}^-(p)$ into a family

    $$\mathcal{S}_v, \qquad 0 < v \leq \delta$$

    of Riemannian submanifolds, each diffeomorphic to $\mathbb{S}^2$.
  - In particular, $\mathcal{S}_\delta$ is the lower boundary of $\mathcal{N}^-(p)$.
- L: (null) tangent vector field to null generators of $\mathcal{N}^-(p)$.
- $\underline{L}$: conjugate null vector field.

Tensor Fields on $\mathcal{N}^-(p)$

- We will deal with the following types of tensorial quantities on $\mathcal{N}^-(p)$:
  1. *Horizontal tensors*: everywhere tangent to the spheres foliating $\mathcal{N}^-(p)$.
     - Corresponding bundle metric and connection given by those induced on the spheres.
  2. *Extrinsic tensors*: tensors on M, restricted to $\mathcal{N}^-(p)$.
     - Corresponding bundle metric and connection given by g.
  3. *Mixed tensors*: generated by tensor products of horizontal and extrinsic tensors.
     - Bundle metric and connection induced accordingly.

Horizontal Tensor Fields

- *Ricci coefficients*: connection coefficients on $\mathcal{N}^-(p)$ that describe its geometry.

$$\chi(X,Y) = g(\overline{D}_X L, Y), \qquad \underline{\chi}(X,Y) = g(\overline{D}_X \underline{L}, Y),$$
$$\zeta(X) = \frac{1}{2} g(\overline{D}_X L, \underline{L}), \qquad \underline{\eta}(X) = g(X, \overline{D}_L \underline{L}).$$

- *Modified mass aspect function*:

$$\mu = \nabla\!\!\!/^a \zeta_a - \frac{1}{2}\hat{\chi}^{ab}\hat{\underline{\chi}}_{ab} + |\zeta|^2 + \frac{1}{4}R_{L\underline{L}L\underline{L}} - \frac{1}{2}R_{L\underline{L}}.$$

Extrinsic Tensor Fields

- Restrictions of g, R, Φ, Ψ, P to $\mathcal{N}^-(p)$.
- Solutions $A^{\mathbf{m}}$ to system of transport equations.
  - $A^{\mathbf{m}}$ has same rank as $\Phi_{\mathbf{m}}$.
  - $f \cdot A^{\mathbf{m}}$ has initial value $J_p^{\mathbf{m}}$ at p, where $J_p^{\mathbf{m}}$ is a tensor at p of the same rank.
  - $A^{\mathbf{m}}$ satisfies the following coupled system of transport equations along the null generators of $\mathcal{N}^-(p)$:

$$\overline{D}_L A^{\mathbf{m}} = -\frac{1}{2}(\operatorname{tr}\chi)A^{\mathbf{m}} + \frac{1}{2}\sum_{\mathbf{c}=1}^{\mathbf{n}} P_{\mathbf{c}}{}^{\mathbf{m}} \cdot A^{\mathbf{c}}.$$

    - Precise indices removed for notational clarity.
  - Note that the first-order terms of our wave equation are handled by altering the $A^{\mathbf{m}}$'s.

Mixed Tensor Fields

- Horizontal derivatives of extrinsic tensor fields form mixed tensor fields.
  - Example: $\overline{\not\Delta} A^{\mathbf{m}}$ - the "mixed horizontal Laplacian" of $A^{\mathbf{m}}$.
- This formalism justifies integration by parts operations needed in the proofs of KSP and GKSP.
- The formalism also shows how the derivation of GKSP can be directly extended to vector bundles.

GKSP, More Detailed Version

- GKSP can be stated more precisely as

$$4\pi \cdot \sum_{\mathbf{m}=1}^{\mathbf{n}} g(\Phi_{\mathbf{m}}|_p, J_p^{\mathbf{m}})$$
$$= \sum_{\mathbf{m}=1}^{\mathbf{n}} \int_{\mathcal{N}^-(p)} g(A^{\mathbf{m}}, \Psi_{\mathbf{m}})$$
$$+ \int_{\mathcal{N}^-(p)} Err(\chi, \underline{\chi}, \zeta, \underline{\eta}, \mu, A, \Phi, P, g, R)$$
$$+ \int_{\mathcal{S}_\delta} Init(\underline{\chi}, A, \Phi, P, g).$$

- For precise (but long) statement, see paper.

A Simplified Setting

- For convenience, we simplify our setting.
  - Assume $\mathbf{n} = 1$, i.e., only one wave equation.
  - Assume no first-order terms.
- Our simplified wave equation:

$$\Box_g \Phi = \Psi \qquad \text{(the setting of KSP).}$$

- Proof of general case follows from similar ideas.

Proof Outline

- Begin with the quantity:

$$\int_{\mathcal{N}^-(p;\epsilon)} g(A, \Psi) = \int_{\mathcal{N}^-(p;\epsilon)} g(A, \Box_g \Phi),$$

where $\mathcal{N}^-(p;\epsilon)$ is the portion of $\mathcal{N}^-(p)$ with $f > \epsilon$.

1. Decompose $\Box_g$ into mixed covariant derivatives.
2. Integrate by parts: move covariant derivatives tangent to $\mathcal{N}^-(p)$ from Φ to A.
3. Let $\epsilon \searrow 0$; boundary terms on $\mathcal{S}_\epsilon$ converge to

$$4\pi \cdot g(\Phi|_p, J_p).$$

Step 1: Decomposition of $\Box_g$

- Goal: Express $\Box_g \Phi$, i.e., two covariant spacetime derivatives of Φ, in terms of mixed covariant derivatives.

$$\Box_g \Phi = \overline{\not\Delta}\Phi - \overline{\not\nabla}_L(\overline{D}_{\underline{L}}\Phi) + 2\underline{\eta} \cdot \overline{\not\nabla}\Phi - \frac{1}{2}(\operatorname{tr}\underline{\chi})\overline{\not\nabla}_L\Phi - \frac{1}{2}(\operatorname{tr}\chi)\overline{D}_{\underline{L}}\Phi + \frac{1}{2}R_{L\underline{L}}[\Phi].$$

- Mixed covariant derivatives are covariant derivatives on $\mathcal{N}^-(p)$, only in directions tangent to $\mathcal{N}^-(p)$.
  - Convenient for integration by parts.

Step 2: Integrations by Parts

- Next, integrate by parts to move mixed derivatives $\overline{\nabla\!\!\!/}$ and $\overline{\nabla\!\!\!/}_L$ from Φ to A.
  - Derivatives $\overline{\nabla\!\!\!/}$ in spherical directions transfer directly.
  - Derivatives $\overline{\nabla\!\!\!/}_L$ in the tangent null direction yield "boundary terms" – integrals over top boundary $\mathcal{S}_\epsilon$ and bottom boundary $\mathcal{S}_\delta$.
- The bottom boundary terms (on $\mathcal{S}_\delta$) form the initial value terms in GKSP.

The Transport Equation

- After integrations by parts, we have the following integrals over $\mathcal{N}^-(p;\epsilon)$:

$$\int_{\mathcal{N}^-(p;\epsilon)} X \cdot \Phi, \qquad \int_{\mathcal{N}^-(p;\epsilon)} Y \cdot \overline{D}_{\underline{L}} \Phi.$$

  - We want to get rid of terms involving $\overline{D}_{\underline{L}} \Phi$.

- However, Y is precisely the transport equation for A and hence vanishes!

Step 3: The Vertex Limit

- Finally, take the limit $\epsilon \searrow 0$.
- Integrals over $\mathcal{N}^-(p; \epsilon)$ become integrals over $\mathcal{N}^-(p)$.
  - These are the fundamental solution and error terms.
- Integrals over $\mathcal{S}_\epsilon$ converge to

$$4\pi \cdot g(\Phi|_p, J_p).$$

  - Φ converges to $\Phi|_p$.
  - fA converges to J_p.
  - Ricci coefficients converge to their Minkowski values.
  - $\mathcal{S}_\epsilon$ "converges to $\mathbb{S}^2$".

The End

Thank you!