Unique Continuation, Carleman Estimates, and Blow-up for Nonlinear Waves

Arick Shao

(joint work with Spyros Alexakis)

Imperial College London

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Outline

Introduce two problems for nonlinear wave equations:

1. Formation of singularities:
   - What happens near a point where a solution blows up?
2. Unique continuation from infinity:
   - Does appropriate "data at infinity" determine a solution?

Survey recent results from Problem (2).

- New global, nonlinear Carleman estimates.

Apply tools from Problem (2) to prove results regarding Problem (1).

Section 1

Formation of Singularities

Nonlinear Wave Equations

Consider the usual model nonlinear wave equations (NLW):

$$\Box\phi + \mu|\phi|^{p-1}\phi = 0, \qquad \Box := -\partial_t^2 + \Delta_x, \quad p > 1.$$

- $\mu = -1$: *defocusing*
- $\mu = +1$: *focusing*

Useful model nonlinear problem—forces dilation symmetry:

- If $\phi(t, x)$ is a solution, then so is

$$\phi_\lambda(t, x) := \lambda^{-\frac{2}{p-1}} \cdot \phi(\lambda^{-1} t, \lambda^{-1} x), \qquad \lambda > 0.$$

- Often determines the appropriate spaces for solving the equation.

Local Well-Posedness

For p not too large (i.e., *energy-subcritical*), there is a standard local well-posedness theory in the energy space:

Theorem (Local Well-Posedness)

Suppose $1 < p < 1 + 4/(n-2)$*. The Cauchy problem with initial data*

$$\phi|_{t=0} = \phi_0 \in H^1(\mathbb{R}^n), \qquad \partial_t \phi|_{t=0} = \phi_1 \in L^2(\mathbb{R}^n),$$

is locally well-posed (i.e., existence of local-in-time solution, uniqueness, continuous dependence on initial data).

Furthermore, the time T *of existence depends on* $\|(\phi_0, \phi_1)\|_{H^1 \times L^2}$.

Global Well-Posedness

Corollary (Continuation Criterion)

If ϕ, as before, exists up to time $0 < T_+ < \infty$, but not at T_+, then

$$\limsup_{t \nearrow T_+} \|(\phi(t), \partial_t \phi(t))\|_{H^1 \times L^2} = \infty.$$

Moreover, NLW arises from a Hamiltonian, hence has conserved "energy":

$$E(t) = \int_{\mathbb{R}^n} \left[\frac{1}{2} |\nabla_{t,x} \phi(t)|^2 - \frac{\mu}{p+1} |\phi(t)|^{p+1} \right] dx.$$

- For the defocusing case, this implies global well-posedness.
- For the focusing case, global well-posedness only for small data.

Blow-Up for Focusing NLW

Simple examples of blow-up come from assuming ϕ depends only on t:

$$\phi_*(t,x) := \left[\frac{2(p+1)}{p-1} \right]^{\frac{1}{p-1}} \cdot (-t)^{\frac{-2}{p-1}} .$$

- For examples with finite energy: localize initial data, and use finite speed of propagation.
- Can also apply Lorentz transforms of ϕ_*.

Question

Generically, what happens when a solution blows up?

Maximal Solutions

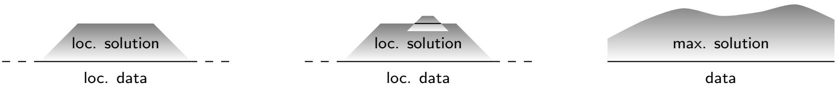
Wave equations obey finite speed of propagation:

- There is an analogous local well-posedness theory in $H^1_{loc} \times L^2_{loc}$.

Can solve equation with initial data on a ball.

- Again, only obstruction is the (local) $H^1 \times L^2$-norm blowing up.

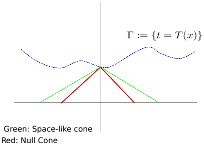
"Solving starting from every possible ball" yields the *maximal solution*.



The Blow-up Graph

One can show the upper boundary of the maximal solution forms a graph $\Gamma = \{(\mathcal{T}(x), x) \mid x \in \mathbb{R}^n\}$.

- Γ is 1-Lipschitz: $|\mathcal{T}(x) - \mathcal{T}(y)| \leq |x - y|$.



$(\mathcal{T}(x_0), x_0) \in \Gamma$ is *noncharacteristic* iff there is a past spacelike cone from $(\mathcal{T}(x_0), x_0)$,

$$\mathcal{C} := \{(t, x) \mid 0 \leq \mathcal{T}(x_0) - t \leq c|x - x_0|\}, \quad c < 1,$$

such that Γ intersects $\mathcal{C}$ only at $(\mathcal{T}(x_0), x_0)$.

Otherwise, $(\mathcal{T}(x_0), x_0)$ is called *characteristic*.

The Case $n = 1$

When $n = 1$, the question was fully answered:

- The family $\mathcal{K}$ of ODE blow-ups ϕ_* and their symmetries is universal.

Theorem (Merle, Zaag; 2007)

Suppose $(0, 0) \in \Gamma$*.*

- *If* $(0, 0)$ *is noncharacteristic, then near* $(0, 0)$*, solution approaches some element of* $\mathcal{K}$*.*
- *If* $(0, 0)$ *is characteristic, then near* $(0, 0)$*, solution approaches a sum of elements in* $\mathcal{K}$*.*

A generalization to higher dimensions fails, because there is no classification of stationary solutions.

Higher Dimensions

In general dimensions, one still has bounds on rate of blow-up.

Theorem (Merle, Zaag; 2005)

Let $1 < p < 1 + 4/(n-1)$*, and suppose* $(0,0) \in \Gamma$*.*

- *If* $(0,0)$ *is noncharacteristic, then* $\exists\, \varepsilon > 0$ *such that* $\forall\ 0 < t \ll 1$*,*

$$\varepsilon \leq t^{\frac{2}{p-1} - \frac{n}{2}} \|\phi(-t)\|_{L^2(B(0,t))} + t^{\frac{2}{p-1} - \frac{n}{2} + 1} \|\nabla_{t,x}\phi(-t)\|_{L^2(B(0,t))} .$$

- *Moreover, given any* $\sigma \in (0,1)$*, we have that* $\forall\ 0 < t \ll 1$*,*

$$t^{\frac{2}{p-1} - \frac{n}{2}} \|\phi(-t)\|_{L^2(B(0,\sigma t))} + t^{\frac{2}{p-1} - \frac{n}{2} + 1} \|\nabla_{t,x}\phi(-t)\|_{L^2(B(0,\sigma t))} \leq K_\sigma .$$

Remark: The blow-up rate matches that of the ODE examples ϕ_*.

The Main Question

Although we know the rate of blow-up (for noncharacteristic points), we do not yet know how blow-up occurs.

Question

If $(0, 0) \in \Gamma$, can one give more information about what is occurring inside the past null cone $\mathcal{N} := \{(-t, x) \mid 0 \leq t \leq |x - x_0|\}$?

Short answer: A significant portion of the H^1-norm within $\mathcal{N}$ must be situated near $\mathcal{N}$ (and cannot be entirely situated in a smaller time cone).

Section 2

Unique Continuation from Infinity

Problem Statement

Question

Consider a linear wave, i.e., solution of

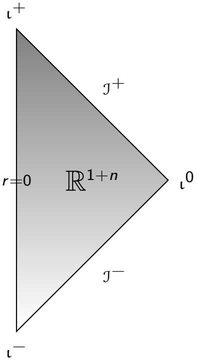
$$\Box\phi + a^{\alpha} D_{\alpha}\phi + V\phi = 0.$$

To what extent does "data" for ϕ at "infinity" (i.e., radiation field) determine ϕ near infinity?

- *Does "vanishing at infinity" imply vanishing near infinity?*

Remark: Could also apply to NLW ($V := \mu|\phi|^{p-1}$).

Minkowski Infinity



Compactified Minkowski,

modulo spherical symmetry.

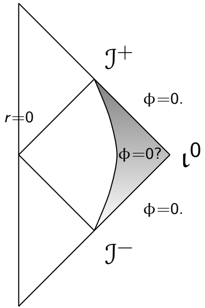
Infinity can be explicitly constructed via *Penrose compactification*.

- Conformally compress "distances":

$$\tilde{g}_M = (1 + |t - r|^2)^{-1}(1 + |t + r|^2)^{-1} g_M.$$

- $(\mathbb{R}^{1+n}, \tilde{g}_M)$ imbeds into *Einstein cylinder*, $\mathbb{R} \times \mathbb{S}^n$.
- Boundary of $\mathbb{R}^{n+1}$ is interpreted as infinity.
- Infinity partitioned into timelike ($\iota^\pm$), spacelike (ι^0), and null ($\mathcal{I}^\pm$) infinities.

(Rough) Theorem Statements



Theorem (Alexakis, Schlue, S.; 2013)

- *Assume* $\Box\phi + V\phi = 0$.
  - V *satisfies asymptotic bounds.*
- *Assume* ϕ *and* $D\phi$ *vanish at least to infinite order on* ι^0 *and half of both* $\mathcal{I}^\pm$.

Then, ϕ *vanishes in the interior near* $\mathcal{I}^\pm$.

Theorem (Alexakis, Schlue, S.; 2014)

Analogous results apply to:

- *Perturbations of Minkowski spacetime.*
- *"Positive-mass spacetimes" (including full Schwarzschild and Kerr families).*

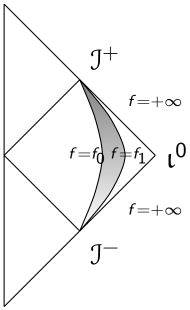
Some Remarks

- Can also handle first-order terms, i.e.,

$$\Box\phi + a^\alpha D_\alpha \phi + V\phi,$$

  if we prescribe vanishing on *more than half* of $\mathcal{I}^\pm$.
- Related results have been established via scattering theory (Friedlander, Sá Barreto, etc.), but assume global solutions on $\mathbb{R}^{1+n}$.
- For "positive mass" spacetimes, all results require vanishing only on arbitrarily small part of $\mathcal{I}^\pm$.

Carleman Estimates



Carleman estimates: main analytical tool in proving unique continuation.

Proposition (Ionescu-Klainerman, Alexakis-Schlue-S.)

Define the function $f = \frac{1}{4}(r^2 - t^2)$. *Then, for* $a > 0$ *and* $f_1 > f_0 > 0$ *sufficiently large:*

$$\int_{\{f_0 < f < f_1\}} f^{2a} f^{-1+\varepsilon} \cdot u^2 \lesssim a^{-1} \int_{\{f_0 < f < f_1\}} f^{2a} f \cdot |\Box u|^2 + \int_{\{f = f_0\}} f^{2a} (\dots u \dots) + \int_{\{f = f_1\}} f^{2a} (\dots u \dots),$$

Carleman to Uniqueness I

Standard arguments yield unique continuation from Carleman estimates.

Proposition (Simplified theorem statement)

Suppose $\Box\phi \equiv 0$, *and* ϕ, $D\phi$ *vanish to infinite order at* $f = \infty$. *Then,* $\phi \equiv 0$ *for sufficiently large* f.

Apply estimate to $u = \chi \cdot \phi$, where:

- ϕ solves wave equation.
- χ is a cutoff function vanishing near $f = f_0$.

Boundary term at $f = f_0$ vanishes.

Take limit $f_1 \nearrow \infty \Rightarrow$ boundary term at $f = f_1$ vanishes:

$$\int_{\{f_0 < f < f_1\}} f^{2a} f^{-1+\varepsilon} \cdot \chi^2 \phi^2 \lesssim a^{-1} \int_{\{f_0 < f < f_1\}} f^{2a} f \cdot |\Box(\chi\phi)|^2 .$$

Carleman to Uniqueness II

Suppose $\chi \equiv 1$ when $f > f_{1/2}$. Then,

$$\int_{\{f_{1/2} < f < f_1\}} f^{2a} f^{-1+\varepsilon} \cdot \phi^2 \lesssim a^{-1} \int_{\{f_0 < f < f_{1/2}\}} f^{2a} f \cdot |D\chi D\phi + \Box\chi \cdot \phi|^2 .$$

Comparing values of f, we can drop the f^{2a}-factors:

$$\int_{\{f_{1/2} < f < f_1\}} f^{-1+\varepsilon} \cdot \phi^2 \lesssim a^{-1} \int_{\{f_0 < f < f_{1/2}\}} f |D\chi D\phi + \Box\chi \cdot \phi|^2 .$$

Letting $a \nearrow \infty$ implies $\phi \equiv 0$ when $f > f_{1/2}$:

- This implies infinite-order vanishing requirement for ϕ.

Section 3

Global Nonlinear Carleman Estimates

Infinite-Order Vanishing

Question

Can one remove the infinite-order vanishing assumption?

No, there are counterexamples—when $n = 3$:

- $\phi(t, x) := r^{-1}$ satisfies $\Box\phi \equiv 0$ near infinity.
- Then, any $\phi_k := (\nabla_x)^k \phi$ also satisfies $\Box\phi_k \equiv 0$.
- But, ϕ_k's vanish to arbitrarily high finite order, but are nonzero.

However, these ϕ_k's fail to be regular at $r = 0$:

- Perhaps can do better when ϕ is "sufficiently global".

Finite-Order Vanishing

Note that in the preceding proof:

- Cutoff function χ needed to make ϕ vanish at $f = f_0$.
- Cutoff function $\chi \Rightarrow a \nearrow \infty \Rightarrow$ infinite-order vanishing.

Thus, *if we could do away with χ, then we may be able to assume only finite-order vanishing for ϕ.*

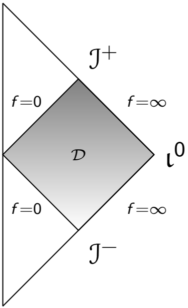
Idea: globalise the Carleman estimate.

- Take $f_0 = 0$, so boundary term $\int_{\{f = f_0\}} f^{2a}(\dots)$ vanishes naturally.

In Minkowski spacetime, this can be done.

- The domain $0 < f < \infty$ is precisely the exterior $\mathcal{D}$ of a null cone.
- Boundary of $\mathcal{D}$ hits origin (where $r = 0$).

A (Rough) Global Result



Theorem (Alexakis, S.; 2014)

- *Assume* $\Box\phi + V\phi = 0$ *in the full exterior* $\mathcal{D}$ *of the null cone about the origin in* $\mathbb{R}^{1+n}$.
  - V *satisfies asymptotic bounds.*
  - V *is sufficiently* L^∞*-small.*
- *Assume* ϕ, $D\phi$ *vanish any power faster than a generic free wave,*

  $$(\text{e.g., } |\phi| \lesssim r^{-\frac{n-1}{2}-\delta} \text{ along null geodesics})$$

  on (exactly) half of $\mathcal{I}^\pm$.

Then, ϕ *vanishes on all of* $\mathcal{D}$.

Small Potentials

Remark: Simple counterexamples show that smallness for V is necessary.

Question

Are there special wave equations for which one does not need smallness of potential for unique continuation?

Consider now the (possibly nonlinear) wave operators

$$\Box' \phi = \Box \phi \pm |\phi|^{p-1} \phi, \qquad p \geq 1.$$

Idea: Derive Carleman estimate for $\Box'$ rather than $\Box$.

- Can we use $\pm |\phi|^{p-1} \phi$ to improve the estimate?

Nonlinear Carleman Estimates

$\pm|\phi|^{p-1}\phi$ generates additional positive (good) terms if:

- Defocusing $(-)$ NLW, $p \geq 1 + 4/(n-1)$.
- Focusing $(+)$ NLW, $p < 1 + 4/(n-1)$.

Proposition (Alexakis, S.)

For the above NLW, the following Carleman estimate holds:

$$\int_{\{0<f<f_1\}} f^{2a} \cdot |\phi|^{p+1} \lesssim a^{-1} \int_{\{0<f<f_1\}} f^{2a} \cdot f|\Box'\phi|^2 + \int_{\{f=f_1\}} f^{2a}(\dots\phi\dots).$$

Remark: Generalizes to NLW of the form $\Box\phi \pm V|\phi|^{p-1}\phi$, if V satisfies certain monotonicity conditions.

Nonlinear Results

Theorem (Alexakis, S.; 2014)

- *Consider on $\mathcal{D}$ solutions of the wave equations,*

$$\Box\phi + V|\phi|^{p+1}\phi = 0, \qquad 1 \leq p < 1 + \frac{4}{n-1},$$
$$\Box\phi - V|\phi|^{p+1}\phi = 0, \qquad p \geq 1 + \frac{4}{n-1},$$

  where $0 < V \in L^\infty$ satisfies certain monotonicity properties.

- *Assume ϕ, $D\phi$ vanish any power faster than usual on half of $\mathcal{I}^\pm$.*

Then, ϕ vanishes on all of $\mathcal{D}$.

Remark: In particular, theorem holds when $V \equiv 1$.

Section 4

Application to Singularity Formation

Nonlinear Carleman Estimates

We return to the subconformal focusing NLW:

$$\Box\phi + |\phi|^{p-1}\phi = 0, \qquad 1 < p < 1 + \frac{4}{n-1}.$$

The nonlinear Carleman estimate yields

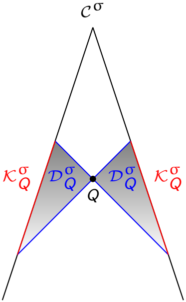
$$\int_{\{0<f<f_1\}} f^{2a} \cdot |\phi|^{p+1} \lesssim \int_{\{f=f_1\}} f^{2a}(\dots\phi\dots).$$

- Estimate has no boundary term on null cone $\{f = 0\}$.

Idea: We replace region of integration $\{0 < f < f_1\}$ by something else?

- Move boundary $\{f = f_1\}$ elsewhere.

A Time Cone Estimate I



Consider the timecone

$$\mathcal{C}^\sigma := \{(-t, x) \mid 0 < r < \sigma t\}, \qquad \sigma \in (0, 1).$$

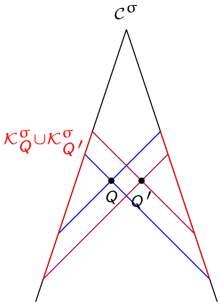
- Consider regions $\mathcal{D}_Q^\sigma$ and $\mathcal{K}_Q^\sigma$ as in the figure.

Then, the nonlinear Carleman estimate yields:

$$\int_{\mathcal{D}_Q^\sigma} f_Q^{2a} |\phi|^{p+1} \lesssim \int_{\mathcal{K}_Q^\sigma} f_Q^{2a} (\dots \phi \dots).$$

- f_Q: translates of f by Q.
- Controls integral of ϕ within $\mathcal{C}^\sigma$ purely by values of ϕ on $\partial \mathcal{C}^\sigma$, not in the interior.

A Time Cone Estimate II



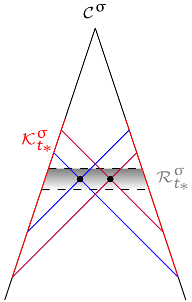
The weight f_Q vanishes at Q:

- No control for ϕ at Q.

Idea: Suppose $t(Q) = t(Q') = t_*$, and sum two such estimates at two separate points Q, Q'.

$$\int_{\mathcal{D}^\sigma_Q} f_Q^{2a} |\phi|^{p+1} + \int_{\mathcal{D}^\sigma_{Q'}} f_{Q'}^{2a} |\phi|^{p+1}$$
$$\lesssim \int_{\mathcal{K}^\sigma_Q} f_Q^{2a} (\dots) + \int_{\mathcal{K}^\sigma_{Q'}} f_{Q'}^{2a} (\dots)$$
$$\lesssim |t_*|^{4a} \int_{\mathcal{K}^\sigma_Q \cup \mathcal{K}^\sigma_{Q'}} (\dots).$$

A Time Cone Estimate III



Now, $\mathcal{D}_Q \cup \mathcal{D}_{Q'}$ contains a slab $\mathcal{R}^\sigma_{t_*}$ on which

$$f_Q + f_{Q'} \gtrsim t_* .$$

Thus, letting $\mathcal{K}^\sigma_{t_*}$ be a large enough slab on $\partial\mathcal{C}$:

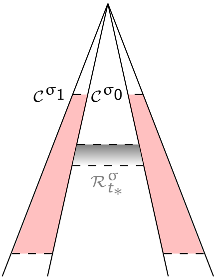
$$|t_*|^{4a} \int_{\mathcal{R}^\sigma_{t_*}} |\phi|^{p+1} \lesssim |t_*|^{4a} \int_{\mathcal{K}^\sigma_{t_*}} (\dots).$$

Proposition

The following estimate holds:

$$\int_{\mathcal{R}^\sigma_{t_*}} |\phi|^{p+1} \lesssim \int_{\mathcal{K}^\sigma_{t_*}} (\dots).$$

A Bulk Estimate



Integrate the cone angle σ over $[\sigma_0, \sigma_1] \subseteq (0, 1)$:

Proposition

The following estimate holds:

$$\int_{\mathcal{R}^{\sigma_0}_{t_*}} |\phi|^{p+1} \lesssim \sup_{t' \simeq t_*} \int_{\sigma_0 |t'| < r < \sigma_1 |t'|} (|\nabla_{t,x} \phi|^2 + t'^{-2} \phi^2)|_{t = t'}.$$

Using Hölder and energy-type estimates, we can bound

$$\int_{\mathcal{R}^{\sigma_0}_{t_*}} (|\nabla_{t,x} \phi|^2 + t_*^{-2} \phi^2).$$

The Main Theorem

Theorem (Alexakis, S.; 2014)

Suppose $\phi \in C^2$ *solves*

$$\Box\phi + |\phi|^{p-1}\phi, \qquad 1 < p < 1 + \frac{4}{n-1},$$

and suppose ϕ *blows up at* $(0,0)$*. If*

$$\limsup_{t_* \nearrow 0} |t_*|^{2-n+\frac{4}{p-1}} \int_{\sigma_0|t_*|<r<\sigma_1|t_*|} (|\nabla_{t,x}\phi|^2 + t_*^{-2}\phi^2)|_{t=t_*} < \delta,$$

then

$$\limsup_{t_* \nearrow 0} |t_*|^{1-n+\frac{4}{p-1}} \int_{\mathcal{R}_{t_*}^{\sigma_0}} (|\nabla_{t,x}\phi|^2 + t_*^{-2}\phi^2) \lesssim \delta.$$

Some Remarks

1. The weights in the estimates correspond to those in the Merle-Zaag bounds (which correspond to the ODE blow-up solutions).
2. The H^1-norm cannot concentrate entirely within a past timecone from $(0, 0)$.
3. The theorem applies to all blow-up points, characteristic and noncharacteristic.
4. Theorem generalizes to NLW of the form

$$\Box\phi + V|\phi|^{p-1}\phi, \qquad 1 < p < 1 + \frac{4}{n-1}, \quad V \simeq 1.$$

Distribution of H^1-Norms

Corollary

Let ϕ and p be as before. If

$$\limsup_{t_* \nearrow 0} |t_*|^{2-n+\frac{4}{p-1}} \int_{r<\sigma_0|t_*|} (|\nabla_{t,x}\phi|^2 + t_*^{-2}\phi^2)|_{t=t_*} > 0,$$

then

$$\limsup_{t_* \nearrow 0} |t_*|^{2-n+\frac{4}{p-1}} \int_{\sigma_0|t_*|<r<\sigma_1|t_*|} (|\nabla_{t,x}\phi|^2 + t_*^{-2}\phi^2)|_{t=t_*} > 0.$$

In other words, some action must be happening near the null cone.

Thank you for your attention!