Unique Continuation from Infinity for Linear Waves

Arick Shao

(joint work with Spyros Alexakis and Volker Schlue)

Imperial College London

Section 1

Introduction

Problem Statement

Problem

- *Consider a linear wave, i.e., solution of*

$$L_g \phi := \Box_g \phi + a^\alpha D_\alpha \phi + V\phi = 0.$$

- *To what extent does "data" for ϕ at infinity (i.e., radiation field) determine ϕ near infinity?*
    - *Does "vanishing at infinity" imply vanishing near infinity?*
- *How does the geometry of the spacetime impact the answer?*
    - *Waves on various asymptotically flat spacetimes.*

Minkowski Infinity

What exactly do we mean by "infinity"?

$\mathbb{R}^{1+n}$: infinity explicitly constructed via *Penrose compactification*.

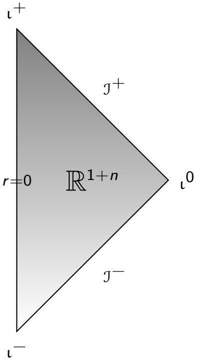
- Compress "distances" via conformal transformation:

$$\tilde{g}_M = \Omega^2 g_M, \qquad \Omega = (1 + |t - r|^2)^{-\frac{1}{2}} (1 + |t + r|^2)^{-\frac{1}{2}}.$$

- $(\mathbb{R}^{1+n}, \tilde{g}_M)$ imbeds into the *Einstein cylinder*, $\mathbb{R} \times \mathbb{S}^n$.
- Boundary of $\mathbb{R}^{1+n}$ interpreted as infinity.

This model is useful for capturing wave propagation.

Asymptotic Flatness



Compactified Minkowski spacetime,

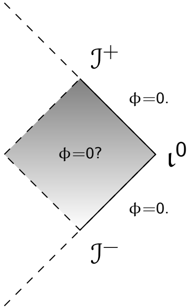
modulo spherical symmetry.

Minkowski infinity partitioned into timelike ($\iota^\pm$), spacelike (ι^0), and null ($\mathcal{I}^\pm$) infinities.

- Describes where geodesics terminate.

More generally, we consider "asymptotically flat" spacetimes, in which one "has a qualitatively analogous model of infinity."

(Rough) Theorem Statement



Theorem

- *Assume* $L_g \phi := \Box_g \phi + a^\alpha D_\alpha \phi + V \phi = 0$.
  - a^α, V *satisfies asymptotic bounds.*
- *Assume* (M, g) *is:*
  - *Perturbation of Minkowski spacetime.*
  - *"Positive-mass spacetime" (including Schwarzschild and Kerr families).*
- *Assume* ϕ *vanishes at least to infinite order on part of null infinity* ($\mathcal{I}^\pm$).

Then, ϕ *vanishes in the interior near* $\mathcal{I}^\pm$.

Some Remarks

- Linear wave equation can be replaced by an *inequality*:

$$|\Box_g \phi| \leq |a||D\phi| + |V||\phi|.$$

- Important feature: applicable to *nonlinear* wave equations.
  - Previous example: general relativity and black hole uniqueness (Alexakis-Ionescu-Klainerman).
- Hyperbolic analogue of "unique continuation from infinity" problem for time-independent Schrödinger operators $-\Delta - V$ (Meshkov, etc.).

Problems in Relativity

- Must time-periodic solutions of Einstein's equations be stationary?
    - Can be related to unique continuation for waves at infinity.
    - Past results (Papapetrou, Bičák-Scholtz-Tod) required analyticity.
- *Inheritance of symmetry*: must matter fields coupled to Einstein equations inherit the symmetries of the spacetime?
    - Stationary spacetimes, various matter models (Bičák-Scholtz-Tod)
    - Counterexamples: Klein-Gordon (Bizoń-Wasserman)
- **Goal:** Eliminate analyticity assumption.

Section 2

Background

Unique Continuation

When we do not have *existence* of solutions, can we still attain *uniqueness*?

Problem (Unique continuation (UC))

Assume the following:

- $p(x, D)$—*linear second-order differential operator on domain* $\mathcal{D} \subseteq \mathbb{R}^m$.
- ϕ—*solution on* $\mathcal{D}$ *of* $p(x, D)\phi \equiv 0$.
- Σ—*hypersurface in* $\mathcal{D}$.

If ϕ *and* $d\phi$ *vanish on* Σ, *then must* ϕ *necessarily vanish (locally) on one side of* Σ?

Elliptic Equations

UC across Σ always holds (Calderón, etc.).

Problem (Strong unique continuation (SUC))

Replace Σ by a point P:

- *If ϕ, $d\phi$ vanish at P, then does ϕ also vanish near P?*

- (Carleman, Aronszajn, Cordes) One now requires *infinite-order vanishing* of ϕ at P, i.e.,

$$\int_{B(P,\delta)} |\phi|^2 r^{-N} < \infty, \qquad r(x) = |x - P|.$$

Hyperbolic Equations

In this case, UC no longer always holds.

(Hörmander) Main criterion for UC for $L_g = \Box_g + a^\alpha D_\alpha + V$ is *pseudoconvexity* of Σ.

- If $\Sigma := \{f = 0\}$ is pseudoconvex (w.r.t. $\Box_g$ and direction of increasing f), then UC for L_g holds from Σ to $\{f > 0\}$.
- (Alinhac) If Σ is not pseudoconvex, then there is an L_g for which UC does not hold across Σ.

Pseudoconvexity

For wave equations, pseudoconvexity can be defined geometrically:

Definition

$\Sigma := \{f = 0\}$ is *pseudoconvex* (w.r.t. $\Box_g$ and increasing f) iff on Σ,

$$D^2 f(X, X) < 0, \text{ if } g(X, X) = Xf = 0.$$

- $-f$ is convex with respect to tangent null directions.
- Any null geodesic that hits Σ tangentially will lie in $\{f < 0\}$ nearby.

Carleman Estimates

Carleman estimates: main tool in proving UC.

- For wave equations, roughly of the form

$$\| e^{-\lambda F(f)} \cdot \Box_g \phi \|_{L^2}^2 \gtrsim \lambda \| e^{-\lambda F(f)} \cdot D\phi \|_{L^2}^2 + \lambda^3 \| e^{-\lambda F(f)} \cdot \phi \|_{L^2}^2 . \qquad (1)$$

  - $\lambda \gg 1$ is a constant.
  - $F(f)$ is a *reparametrization* of f (e.g., $\log f$).
- By standard arguments, (1) implies UC for $\Box_g$.

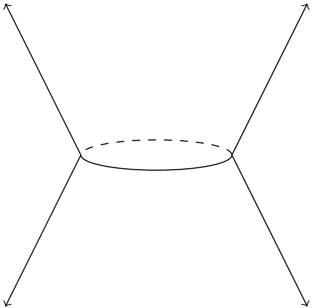
Example: Bifurcate Null Cones

- Consider a *bifurcate null cone* in Minkowski space, e.g.,

$$\Sigma = \mathcal{N}_{r_0} := \{|t| = |r| - r_0\} \subseteq \mathbb{R}^{n+1}.$$

- (Ionescu-Klainerman): Unique continuation from $\mathcal{N}_{r_0}$ to outer region.
- Applications: black hole uniqueness results (Alexakis-Ionescu-Klainerman).

Question: What happens when $r_0 \searrow 0$.



Bifurcate null cone.

Hyperbolic SUC

What is a hyperbolic analogue for SUC?

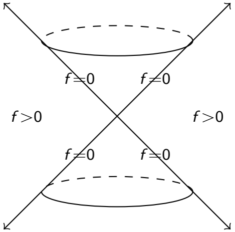
- Elliptic ($\mathbb{R}^n$): (∞-order) vanishing at $r^2 = 0 \Rightarrow$ vanishing on $r^2 \ll 1$.

$$r^2 = |x|^2 = (x^1)^2 + \cdots + (x^n)^2.$$

- Hyperbolic ($\mathbb{R}^{1+n}$): replace r^2 by

$$f = (x^1)^2 + \cdots + (x^n)^2 - (x^0)^2 = r^2 - t^2.$$

  - Vanishing at $f = 0 \Rightarrow$ vanishing for $0 < f \ll 1$?
  - This is UC from null cone to exterior.



Null cone.

The Minkowski Case

Lemma (Ionescu-Klainerman)

Assume:

- ϕ *satisfies* $\Box\phi + V\phi = 0$.
  - V *satisfies certain decay assumptions.*
- ϕ *vanishes to infinite order on the null cone* $\mathcal{N}_0 := \{f = 0\}$.

Then, ϕ *vanishes in the region* $0 < f \ll 1$.

Remark: No first-order terms allowed in wave equation.

- Because level sets of f have exactly zero pseudoconvexity.

As before, proof is via a Carleman estimate.

General Cases

(Alexakis-Schlue-S.) New extensions of previous result:

1. Generalizations of vanishing assumptions.
   - If we prescribe *exponential*, and not just ∞-order, vanishing at $\mathcal{N}_0$, then the UC theorem applies to a wider class of V.
   - In general: correspondence between vanishing condition for ϕ and wave operators $\Box + V$ for which theorem holds.
2. *Geometric robustness:* extensions to many non-flat metrics.
   - Main idea: refined Carleman estimates, proved using entirely geometric methods (covariant derivatives, integration by parts).

Geometric Robustness

Lemma

- *Lorentz metric g, given in "almost null coordinates",*

$$\bar{u} \approx t - r, \qquad \bar{v} \approx t + r.$$

- *Level sets of $f := -\bar{u}\bar{v}$ are pseudoconvex.*
- *ϕ vanishes at least to ∞-order at $\mathcal{N}_0 := \{f = 0\}$.*
- *Some other technical conditions relating g and pseudoconvexity.*

Then, ϕ also vanishes on $0 < f \ll 1$.

If pseudoconvexity is positive, then first-order terms allowed in wave equation (i.e. $\Box_g + a^\alpha D_\alpha + V$).

Section 3

Unique Continuation from Infinity

The Conformal Inversion

- Consider first Minkowski spacetime, $\mathbb{R}^{1+n}$, with

$$g_M = -4dudv + r^2 \mathring{\gamma}.$$

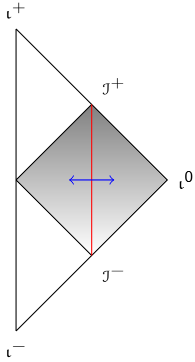
- Recall the conformal inversion,

$$\Psi(\xi) := \frac{c\xi}{g_M(\xi, \xi)}.$$

  - Ψ is a conformal isometry:

$$\Psi^* g_M = (uv)^{-2} \cdot g_M = f^{-2} g_M.$$

  - Identifies half of $\mathcal{I}^+ \cup \mathcal{I}^-$ with $\mathcal{N}_0$.



A Preliminary Result

Lemma

Assume:

- ϕ *vanishes to infinite/exponential order on half of* $\mathcal{I}^+ \cup \mathcal{I}^-$.
- ϕ *satisfies* $\Box\phi + V\phi = 0$, *and, near infinity,*

$$V \in \mathcal{O}((|u||v|)^{-1-\varepsilon}) \quad / \quad V \in \mathcal{O}(1).$$

Then, ϕ *vanishes near infinity.*

What about wave equations with first-order terms?

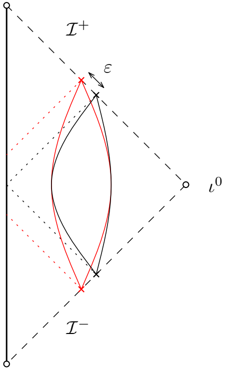
- For this, we must find some pseudoconvexity.

Finding Pseudoconvexity

- Consider "a bit more than half of null infinity":

$$\mathcal{I}_\varepsilon := \{v = \infty,\ u < \varepsilon\} \cup \{u = -\infty,\ v > -\varepsilon\}.$$

- Consider $f_\varepsilon := (-u + \varepsilon)^{-1}(v + \varepsilon)^{-1}$.
- Positive level sets of f_ε are hyperboloids.
    - Level sets focus at boundary of $\mathcal{I}_\varepsilon$.
    - $\{f_\varepsilon = 0\}$ corresponds to $\mathcal{I}_\varepsilon$.
- Level sets $\{f_\varepsilon = c\}$ are *pseudoconvex*.
    - Pseudoconvexity degenerates as $c \searrow 0$.



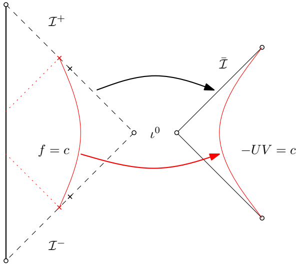
Red lines: level sets of f_ε.
Black lines: null geodesics.

(Figure by V. Schlue.)

A Warped Inversion

While there is no inversion Ψ adapted to f_ε, the idea of a conformal factor survives.

- Construct a "warped" conformal inversion.
  1. Conformal transformation of g_M:
  $$\bar{g}_M := f_\varepsilon^2 \cdot g_M .$$
  2. Change of coordinates:
  $$\bar{u} := -(v+\varepsilon)^{-1}, \qquad \bar{v} := (-u+\varepsilon)^{-1}.$$
- In "inverted" coordinates,
$$\bar{g}_M = -4 d\bar{u} d\bar{v} + f_\varepsilon^2 r^2 \cdot \mathring{\gamma}, \qquad f_\varepsilon = -\bar{u}\bar{v}.$$



Warped inversion of Minkowski.

(Figure by V. Schlue.)

Geometric Robustness, Revisited

This once again looks like hyperbolic SUC.

- Pseudoconvexity is conformally invariant.
    - Thus, level sets of f_ε also pseudoconvex in $\bar{g}_M$.
- While $\bar{g}_M$ is not Minkowski, it satisfies our lemma.
- Since level sets of f_ε are pseudoconvex, we can also treat wave equations with first-order terms.

What if we perturb the Minkowski metric ($g = g_M + \delta$)?

- If δ (in null coordinates) decays fast enough toward $\mathcal{I}_\varepsilon$, then spacetime, after similar inversion, satisfies hyperbolic SUC lemma.
- (These spacetimes have zero mass.)

Main Theorem 1.1

Theorem (Alexakis-Schlue-S., 2013)

Decaying potential case. *Consider a metric g over $\mathbb{R}^{n+1}$ of the form*

$$g = \mu du^2 - 4K du dv + \nu dv^2 + \sum_{A,B=1}^{n-1} r^2 \gamma_{AB} dy^A dy^B + \sum_{A=1}^{n-1} (c_{Au} dy^A du + c_{Av} dy^A dv),$$

with the components satisfying

$$K = 1 + \mathcal{O}_1^\varepsilon(r^{-2}), \qquad \gamma_{AB} = \mathring{\gamma}_{AB} + \mathcal{O}_1^\varepsilon(r^{-1}), \qquad c_{Au}, c_{Av} = \mathcal{O}_1^\varepsilon(r^{-1}), \qquad \mu, \nu = \mathcal{O}_1^\varepsilon(r^{-3}).$$

(Here, $\mathcal{O}_1^\varepsilon(W)$ denotes functions in $\mathcal{O}(W)$ up to first derivatives, with constant $\ll \varepsilon$.) Consider also a wave operator $L_g := \Box_g + a^\alpha D_\alpha + V$, where

$$a^u \in \mathcal{O}((v+\varepsilon)^{-1} r^{-\frac{1}{2}}), \qquad a^v \in \mathcal{O}((-u+\varepsilon)^{-1} r^{-\frac{1}{2}}), \qquad a^I \in \mathcal{O}(f_\varepsilon^{\frac{1}{2}} r^{-\frac{3}{2}}), \qquad V \in \mathcal{O}(f_\varepsilon^{1+\eta}),$$

for some $\eta > 0$. Consider any C^2-solution ϕ of $L_g \phi = 0$, which in addition vanishes at $\mathcal{I}_\varepsilon$ faster than any power of r (in an L^2-sense). Then, ϕ also vanishes near $\mathcal{I}_\varepsilon$.

Main Theorem 1.2

Theorem (Alexakis-Schlue-S., 2013)

Bounded potential case. *Consider* $(\mathbb{R}^{n+1}, g)$ *as before. Consider also any wave operator* $L_g := \Box_g + a^\alpha D_\alpha + V$*, where*

$$a^u \in \mathcal{O}((v+\varepsilon)^{-1} f_\varepsilon^{-\frac{1}{3}} r^{-\frac{1}{2}}), \qquad a^v \in \mathcal{O}((-u+\varepsilon)^{-1} f_\varepsilon^{-\frac{1}{3}} r^{-\frac{1}{2}}), \qquad a^I \in \mathcal{O}(f_\varepsilon^{\frac{1}{6}} r^{-\frac{3}{2}}), \qquad V \in \mathcal{O}(1).$$

Consider any C^2*-solution* ϕ *of* $L_g \phi = 0$*, which in addition vanishes at* $\mathcal{I}_\varepsilon$ *faster than any power of* $\exp(r^{4/3})$ *(in an* L^2*-sense).*

Then, ϕ *also vanishes near* $\mathcal{I}_\varepsilon$*.*

Remarks on Optimality

- The infinite-order vanishing assumptions for ϕ are necessary.
  - At least, when ϕ is locally defined near infinity.
- For first theorem, there are counterexamples with $V \in \mathcal{O}(f_\varepsilon^{1-\eta})$.
- For second theorem, there are counterexamples with $V \in \mathcal{O}(f_\varepsilon^{-\eta})$.
- Do not expect unique continuation from less than half of null infinity (due to argument of Alinhac).

Remark: in contrast to many earlier results (Helgason, Sá Barreto, etc.), we work only locally near infinity, both for assumption and conclusion.

The Schwarzschild Exterior

- Outer region of Schwarzschild spacetime with mass $m > 0$:

$$M := \mathbb{R}_t \times (2m, \infty)_r \times \mathbb{S}^2,$$
$$g_S := -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \mathring{\gamma}.$$

- How does Schwarzschild differ from Minkowski?
    - Minkowski: leading order pseudoconvexity comes from anchor point of the hyperboloids.
    - Schwarzschild: *leading order pseudoconvexity from positive mass.*
- This leads to *stronger* UC results than in Minkowski.

Null Coordinates

- *Tortoise coordinate*: fix $r_0 > 2m$, and let

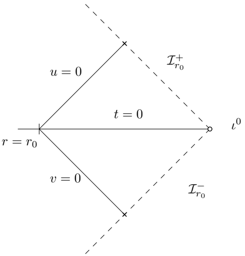
$$r_*(r) := \int_{r_0}^{r} \left(1 - \frac{2m}{s}\right) ds.$$

- Null coordinates then defined by

$$u := \frac{1}{2}(t - r_*), \qquad v := \frac{1}{2}(t + r_*).$$

- In null coordinates,

$$g_S = -4 \left(1 - \frac{2m}{r}\right) dudv + r^2 \mathring{\gamma}.$$



Schwarzschild in null coordinates.

(Figure by V. Schlue.)

Pseudoconvexity in Schwarzschild

- Define $f_{r_0} = -u^{-1} v^{-1}$, whose level sets are hyperboloids which focus at $\{v = \infty,\ u = 0\}$ and $\{u = -\infty,\ v = 0\}$.
  - In particular, anchor points depend on choice of r_0.
- Main observation: level sets of f_{r_0} are pseudoconvex, *regardless of choice of* r_0.
  - Thus, by choosing r_0 large enough, we get unique continuation *from an arbitrarily small part of null infinity* (containing ι^0).

Reduction to Hyperbolic SUC

- We can define an analogous "conformal inversion",

$$\bar{g}_S := \left(1 - \frac{2m}{r}\right)^{-1} f_{r_0}^2 \cdot g_S \,, \qquad \bar{u} := -v^{-1}, \quad \bar{v} := -u^{-1}.$$

- In the inverted coordinates,

$$\bar{g}_S = -4 d\bar{u} d\bar{v} + \left(1 - \frac{2m}{r}\right)^{-1} r^2 \cdot \mathring{\gamma}.$$

- Again, this satisfies the hyperbolic SUC lemma.

Perturbations of Schwarzschild

Geometric robustness: process also works for perturbations of g_S.

- Includes the *entire Kerr family*, after coordinate change.

Theorem (Alexakis-Schlue-S., 2013)

The main theorems for near-Minkowski spacetimes have direct analogues for near-Schwarzschild spacetimes, including all Kerr spactimes. The main difference with the near-Minkowski theorems is the following improvement: (infinite-order) vanishing is required for only an arbitrarily small part of null infinity.

The General Class

Results extend to a general class of dynamical, positive-mass spacetimes.

- Manifold (M, g) given (in almost-null coordinates) by

$$\mathcal{D} := (-\infty, 0)_u \times (0, \infty)_v \times \mathbb{S}^{n-1},$$

$$g := \mu du^2 - 4K du dv + \nu dv^2 + \sum_{A,B=1}^{n-1} r^2 \gamma_{AB} dy^A dy^B + \sum_{A=1}^{n-1} (c_{Au} dy^A du + c_{Av} dy^A dv).$$

  - Similar to near-Minkowski, but we prescribe positive mass.
  - Contains perturbations of Schwarzschild as special case.

Asymptotic Assumptions

- *Metric decay:* The components of g satisfy:

$$K = 1 - \frac{2m}{r}, \qquad \gamma_{AB} = \mathring{\gamma}_{AB} + \mathcal{O}_1\left(\frac{1}{v-u}\right),$$

$$c_{Au}, c_{Av} = \mathcal{O}_1\left(\frac{1}{v-u}\right), \qquad \mu, \nu = \mathcal{O}_1\left(\frac{1}{(v-u)^3}\right).$$

- *Positive mass:* m is a *function* on $\mathcal{m}$ satisfying $m \geq m_{\min} > 0$. Moreover, dm satisfies certain decay estimates.
  - In particular, m has limits at null infinity.
- *Radial function:* r is also a (not necessarily spherically symmetric) function satisfying certain asymptotic assumptions.
  - r and $r_* := v - u$ are related like in Schwarzschild: $r_* - r \simeq \log r$.

Reduction to Hyperbolic SUC

Though more computationally intense, the idea is same as before.

- Level sets of $f := -u^{-1}v^{-1}$ are pseudoconvex.
- Conformal inversion of metric, $\bar{g} := K^{-1}f^2 \cdot g$.

Then, $\bar{g}$ satisfies the hyperbolic SUC lemma.

- In fact, UC results for perturbations of Minkowski, perturbations of Schwarzschild, and this general class are proved all at once.

Main Theorems 2

Theorem (Alexakis-Schlue-S., 2013)

Consider (M, g) *as above. Consider also any wave operator* $L_g := \Box_g + a^\alpha D_\alpha + V$, *where*

$$a^u \in \mathcal{O}(v^{-1} r^{-\frac{1}{2}}), \qquad a^v \in \mathcal{O}((-u)^{-1} r^{-\frac{1}{2}}), \qquad a^I \in \mathcal{O}(f^{\frac{1}{2}} r^{-\frac{3}{2}}), \qquad V \in \mathcal{O}(f^{1+\eta}),$$

for some $\eta > 0$. *Consider any* C^2*-solution* ϕ *of* $L_g \phi = 0$, *which vanishes at* $\mathcal{I} = \{v = \infty, u < 0\} \cup \{u = -\infty, v > 0\}$ *faster than any power of r (in an* L^2*-sense). Then,* ϕ *also vanishes near* $\mathcal{I}$.

Theorem (Alexakis-Schlue-S., 2013)

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Consider any C^2*-solution* ϕ *of* $L_g \phi = 0$, *which vanishes at* $\mathcal{I}$ *faster than any power of* $\exp(r^{4/3})$ *(in an* L^2*-sense). Then,* ϕ *also vanishes near* $\mathcal{I}$.

Conclusions

Roughly, if ϕ vanishes to infinite order at a certain part of the null infinity, then ϕ vanishes in a neighborhood in the physical interior.

1. Connection between pseudoconvexity (PDE and geometric notion) and positive mass (relativistic notion).
2. Connection between unique continuation from infinity and from null cones, via "conformal inversions".
3. View of unique continuation from null cones as hyperbolic analogue of strong unique continuation for elliptic equations.

The End

- Thank you for your attention!

Section 5

Appendix

Statement of the Estimates

Main tool for hyperbolic SUC is Carleman estimate.

- (For our main results, this is the inverted setting.)

General form:

$$\| e^{-\lambda F(f)} \Box_g \phi \|_{L^2}^2 \gtrsim \lambda \sum_\alpha \| e^{-\lambda F(f)} A^\alpha D_\alpha \phi \|_{L^2}^2 + \lambda^3 \| e^{-\lambda F(f)} B \phi \|_{L^2}^2 .$$

- $\lambda \ll 1$.
- $F(f_\varepsilon)$ is a reparametrization of f_ε.
- A^α, B are positive weights that blow up or decay at $\{f = 0\}$.

Proof of Carleman estimate is purely geometric.

Main Ideas

Carleman estimate can be thought of as an energy estimate for $\Box_g$, but:

1. We want boundary terms to vanish.
2. We want bulk terms to be positive.

Objective (1) achieved by:

- Vanishing assumptions for ϕ at $f = 0$.
- Cutoff functions for $f = f_0 > 0$.

Objective (2) achieved using a *positive commutator*.

- Consider wave equation not for ϕ, but for $\psi = \mathcal{F}\phi$.

Positive Commutators

To ensure the bulk term is positive:

1. Bulk terms containing derivative of ϕ tangent to level sets of $\mathcal{F}$:
   - These are positive only when level sets of $\mathcal{F}$ are pseudoconvex.
   - Thus, $\mathcal{F} = f$ is a candidate.
2. Bulk terms containing ϕ and derivative normal to level sets of $\mathcal{F}$:
   - Additional freedom: any reparametrization $F \circ \mathcal{F} = F \circ f$ (where $F' > 0$) produces same level sets.
   - Find reparametrization $F(f)$ so these bulk terms are positive.
   - Many valid choices of F—as long as F grows fast enough.

Some Features

Weights A^α and B depend on pseudoconvexity and on choice of $F(f)$.

- $F(f)$ must grow "at least as fast as $\log f$" (but cannot be log itself).
- (Ionescu-Klainerman) Choose $F = \log f + \text{correction}$.
  - *Decaying potential case*: $|V| \lesssim f^{1+}$, requires ∞-order vanishing of ϕ.
- (New) Choose $F = -f^{-2/3}$.
  - *Bounded potential case*: $|V| \lesssim 1$, requires exponential vanishing of ϕ.

Finite-Order Vanishing

Can we somehow do away with the infinite-order vanishing assumption?

- Cannot do so while remaining local near infinity (counterexamples).
- (Alexakis-S.) Yes on Minkowski spacetime, if we have *global* information for ϕ.

Technical obstruction to finite-order vanishing comes from cutoff function to make boundary terms vanish.

- If we can go from infinity all the way to null cone about origin, then boundary terms vanishing without cutoff function.
- Requires very careful choice of reparametrization of f.

Nonlinear Equations

The finite-order vanishing theorems have a new obstruction:

- Linear potential must also be small.

(Alexakis-S.) However, for some nonlinear equations, we can treat nonlinearity directly within Carleman estimates:

- Focusing, subconformal nonlinearity.
- Defocusing, conformal and superconformal nonlinearity.

In these cases, can eliminate smallness assumption.

(Alexakis-S.) These nonlinear Carleman estimates have other applications:

- Final states.
- Formation of singularities.