Correspondence and Rigidity Results on Asymptotically Anti-de Sitter Spacetimes

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Includes joint work with G. Holzegel (Imperial College London)

Section 1

Introduction

Correspondence and Holography

Outstanding problem in theoretical physics:

- Reconciling Einstein's theory of gravity with quantum field theories.

Influential research direction:

- AdS/CFT correspondence
- (AdS: Anti-de Sitter)
- (CFT: Conformal field theory)

AdS/CFT $\Rightarrow$ holographic principle:

- Gravitational theory on spacetime encoded in some theory on its boundary (of one less dimension).

Original paper*: ~~12154~~ ~~12201~~ ~~12381~~ ~~12869~~ ~~13156~~ ~~13278~~ ~~13964~~ ~~14577~~ 14769 citations.†

- * J. Maldacena, *The large N limit of superconformal field theories and supergravity* (1999)
- † Data from `http://inspirehep.net/record/451647/citations`.

What's Missing?

In AdS context, little rigorous mathematics for:

- Positive statements of this principle.
- Precise formulations of this principle.

In particular, in dynamical (non-stationary) settings.

Main questions:

1. Rigorous statements toward holographic correspondences?
2. Proofs of these statements?
3. Mechanisms behind such correspondences?

General Relativity

Gravity described by Einstein's theory of general relativity:

- Spacetime: $(n+1)$-dimensional Lorentzian manifold $(\mathscr{M}, \mathbf{g})$.
- $\mathbf{g}$: Lorentzian metric, with signature $(-,+,\dots,+)$.

No matter fields $\Rightarrow$ $\mathbf{g}$ satisfies Einstein-vacuum equations (EVE):

$$\mathrm{Ric}_{\mathbf{g}} = \frac{2\Lambda}{n-1}\mathbf{g}.$$

- $\mathrm{Ric}_{\mathbf{g}}$: Ricci curvature of $\mathbf{g}$.
- $\Lambda \in \mathbb{R}$: Cosmological constant.

Anti-de Sitter Spacetime

Anti-de Sitter (AdS) spacetime:

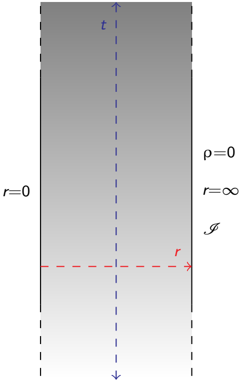
- Maximally symmetric solution of EVE, with $\Lambda = \frac{-n(n-1)}{2}$.
- $\Lambda < 0$ analogue of Minkowski spacetime.
- Lorentzian analogue of hyperbolic space.

Global representation of AdS spacetime:

$$(\mathbb{R}_t \times \mathbb{R}^n_x, \mathbf{g}_0), \qquad \mathbf{g}_0 := (1 + r^2)^{-1} dr^2 - (1 + r^2) dt^2 + r^2 \mathring{\gamma}.$$

- $\mathring{\gamma}$: Round metric for unit sphere $\mathbb{S}^{n-1}$.

The Conformal Boundary



Conformal AdS, mod $\mathbb{S}^{n-1}$.

Consider inverted radius $\rho := r^{-1}$:

$$\mathbf{g}_0 = \rho^{-2} \left[(1+\rho^2)^{-1} d\rho^2 - (1+\rho^2) dt^2 + \mathring{\gamma} \right].$$

- $\rho^2 \mathbf{g}_0$: smooth at $\rho = 0$ ($r = \infty$).

Formally attach boundary at "$\rho = 0$":

$$(\mathscr{I} \simeq \mathbb{R}_t \times \mathbb{S}^{n-1}, \ \mathring{\mathfrak{g}} := -dt^2 + \mathring{\gamma}).$$

- $\mathscr{I}$: conformal boundary of AdS:
- $\mathring{\mathfrak{g}}$: (Lorentzian) boundary metric.

Asymptotically AdS (aAdS):

- Spacetime with "similar conformal boundary".

A Correspondence Question

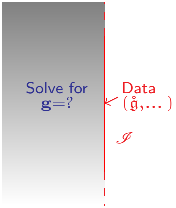
Question (Preliminary)

Is there some one-to-one correspondence between:

- *aAdS solution of EVE ("gravitational dynamics").*
- *Data prescribed at conformal boundary $\mathscr{I}$.*
    - *(Boundary metric $\mathfrak{g}$̊, boundary stress-energy tensor.)*

Attempt 1: Formulate in terms of PDEs.

- *Given:* (Cauchy) data on conformal boundary $\mathscr{I}$.
- *Goal:* Solve EVE into interior?



EVE, with data on $\mathscr{I}$.

Ill-Posedness

Bad news: Problem is ill-posed.

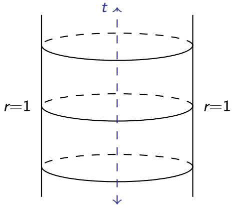
- "Initial" hypersurface $\mathscr{I}$ is timelike!

For wave equations on bounded domain, need:

- Initial data at $t = 0$.
- Dirichlet **or** Neumann data on $\mathcal{C}$.

To solve EVE, need:

- Initial data at $t = 0$.
- Dirichlet **or** Neumann data on $\mathscr{I}$.
- (Friedrich, 1995), (Enciso–Kamran, 2019)



The cylinder $\mathcal{C}$.

A Unique Continuation Problem

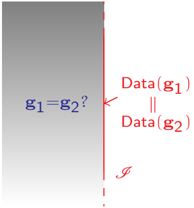
Attempt 2: Formulate as unique continuation (UC) problem.

- Classical problem in PDEs.
- *If a solution exists, then must it be unique?*

Question (Correspondence, Informal)

Given two aAdS solutions $\mathbf{g}_1$, $\mathbf{g}_2$ of EVE:

- *If $\mathbf{g}_1$, $\mathbf{g}_2$ have same boundary data on $\mathscr{I}$, ...*
- *... then must these spacetimes be isometric?*



Section 2

Precise Formulations

aAdS Manifolds

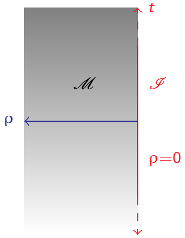
Goal: Precise description of our aAdS spacetimes.

Step 1: Construction of aAdS manifold $\mathscr{M}$.

- Conformal boundary: $\mathscr{I}^n := \mathbb{R}_t \times \mathcal{S}^{n-1}$.
  - $\mathcal{S}$: cross-section of $\mathscr{I}$.
- Spacetime (near boundary): $\mathscr{M} := (0, \rho_0]_\rho \times \mathscr{I}$.

Remark. Formulation allows for:

- General boundary topology/geometry.
- *Example:* AdS, planar AdS, toroidal AdS.



aAdS manifold $\mathscr{M}$, mod $\mathcal{S}$.

aAdS Metrics

Step 2: Construction of aAdS metric g.

- Assume g in Fefferman–Graham (FG) gauge.

$$\mathrm{g} := \rho^{-2} (d\rho^2 + \mathfrak{g}_{ab} \, dx^a dx^b).$$

  - ρ trivial, decoupled from $(t, \mathcal{S})$-coordinates.
  - $\mathfrak{g}$: vertical metric (ρ-indexed family of Lorentzian metrics on $\mathscr{I}$).
- Assume $\mathfrak{g}$ has (Lorentzian) boundary limit:

$$\lim_{\rho \searrow 0} \mathfrak{g} = \mathring{\mathfrak{g}}.$$

Remark. No loss of generality from FG gauge.

Einstein–Vacuum Spacetimes

Q. What if $(\mathscr{M}, g)$ also satisfies EVE?

- What structure does this impose on g at $\mathscr{I}$?

(Fefferman–Graham, 1984, 2007) Ambient metric construction.

- Analytic conformal data on null cone.
- Spacetime given as series expansion in terms of conformal data.
- (Kichenassamy, 2004) Series converges for analytic data.

Idea adapted to aAdS settings:

- **Given:** Analytic conformal boundary data on $\mathscr{I}$.
- **Derive:** Formal series expansion from $\mathscr{I}$ for g.

Fefferman–Graham Expansions

Fefferman–Graham expansion for vertical metric g:

$$\mathsf{g} = \begin{cases} \sum_{k=0}^{\frac{n-1}{2}} \rho^{2k} \mathfrak{g}^{(2k)} + \rho^n \mathfrak{g}^{(n)} + \dots & n \text{ odd}, \\ \sum_{k=0}^{\frac{n}{2}} \rho^{2k} \mathfrak{g}^{(2k)} + \rho^n (\log \rho) \mathfrak{g}^{(*)} + \rho^n \mathfrak{g}^{(n)} + \dots & n \text{ even}. \end{cases}$$

- $\mathfrak{g}^{(0)} = \mathring{\mathfrak{g}}$: freely prescribed (boundary metric).
- $\mathfrak{g}^{(n)}$: also partially free (related to boundary stress-energy tensor).
- Coefficients before $\mathfrak{g}^{(n)}$: determined by $\mathfrak{g}^{(0)}$.
- Coefficients after $\mathfrak{g}^{(n)}$: determined by $\mathfrak{g}^{(0)}$ and $\mathfrak{g}^{(n)}$.

n even $\Rightarrow$ anomalous $\rho^n (\log \rho)$-term.

- Expansion after $\mathfrak{g}^{(n)}$ is polyhomogeneous (includes $\rho^l (\log \rho)^m$).

Non-Analytic Settings

Full FG expansion only applicable to analytic settings.

- Too restrictive.

Q. What about generic spacetimes?

- With finite regularity (H^s, C^M).
- Setting of well-posedness theories of EVE.

Expect. Partial FG expansion (to finite order).

- But, do not wish to assume this a priori.

Partial FG Expansions

Theorem (S., 2019)

Assume $\mathfrak{g}$ *satisfies EVE, and assume*

$$\lim_{\rho \searrow 0} \mathfrak{g} \rightarrow \mathfrak{\mathring{g}}, \qquad \|\mathfrak{g}\|_{C^{n+2}_{\mathscr{I}}(\mathscr{M})} < \infty, \qquad \|\partial_\rho \mathfrak{g}\|_{C^{0}_{\mathscr{I}}(\mathscr{M})} < \infty.$$

Then, $\mathfrak{g}$ *has the following expansion near* $\mathscr{I}$*:*

$$\mathfrak{g} = \begin{cases} \sum_{k=0}^{\frac{n-1}{2}} \rho^{2k} \mathfrak{g}^{(2k)} + \rho^n \mathfrak{g}^{(n)} + o(\rho^n) & n \text{ odd}, \\ \sum_{k=0}^{\frac{n}{2}} \rho^{2k} \mathfrak{g}^{(2k)} + \rho^n (\log \rho) \mathfrak{g}^{(*)} + \rho^n \mathfrak{g}^{(n)} + o(\rho^n) & n \text{ even}. \end{cases}$$

- $\mathfrak{g}^{(0)} = \mathfrak{\mathring{g}}$ *is the boundary metric.*
- *Coefficients between* $\mathfrak{g}^{(0)}$ *and* $\mathfrak{g}^{(n)}$*: determined by* $\mathfrak{g}^{(0)}$*.*
- *Derives existence of free* $\mathfrak{g}^{(n)}$*-term.*

(Chruściel–Delay–Lee–Skinner, 2005) Riemannian analogue.

Remarks and Examples

Remark. $-\mathfrak{g}^{(2)}$ is the Schouten tensor for $\mathfrak{g}$:

$$-\mathfrak{g}^{(2)} = \frac{1}{n-2} \left[\mathring{\mathfrak{Ric}} - \frac{1}{2(n-1)} \mathring{\mathfrak{Sc}} \cdot \mathring{\mathfrak{g}} \right], \qquad n > 2.$$

Example

Schwarzschild-AdS spacetime, with mass $M \in \mathbb{R}$:

$$\mathbf{g}_M := \left(1 + r^2 - \frac{M}{r^{n-2}} \right)^{-1} dr^2 - \left(1 + r^2 - \frac{M}{r^{n-2}} \right) dt^2 + r^2 \mathring{\gamma}, \qquad \mathfrak{g}^{(0)} = -dt^2 + \mathring{\gamma}.$$

$$\mathfrak{g}^{(2)} = \begin{cases} -\frac{1}{2}(dt^2 + \mathring{\gamma}) & n > 2, \\ -\frac{1-M}{2}(dt^2 + \mathring{\gamma}) & n = 2, \end{cases} \qquad \mathfrak{g}^{(n)} = \begin{cases} \frac{M}{n}[(n-1)dt^2 + \mathring{\gamma}] & n \notin \{2, 4\}, \\ \frac{1}{16}(-dt^2 + \mathring{\gamma}) + \frac{M}{4}(3dt^2 + \mathring{\gamma}) & n = 4. \end{cases}$$

Return to Correspondence

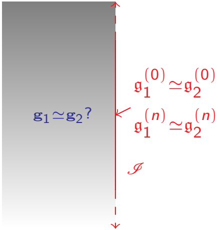
Thus, boundary data for EVE given by:

- $\mathfrak{g}^{(0)} = \mathring{\mathfrak{g}}$: "Dirichlet branch".
- $\mathfrak{g}^{(n)}$: "Neumann branch".

Question (Correspondence)

Given two aAdS solutions g_1, g_2 of EVE:

- *Assume g_1, g_2 have equivalent values for $\mathfrak{g}^{(0)}$, $\mathfrak{g}^{(n)}$.*
- *Must g_1, g_2 be isometric?*
- *Do $\mathfrak{g}^{(0)}$ and $\mathfrak{g}^{(n)}$ determine solution of EVE?*



The correspondence problem.

The State of Affairs

(Biquard, 2008; Anderson–Herzlich, 2008, 2010)

- Analogue in Riemannian/elliptic case.
- If $(\mathscr{N}, \mathbf{h})$ is Riemannian, asymptotically hyperbolic, and Einstein...
- ... then $\mathfrak{h}^{(0)}$, $\mathfrak{h}^{(n)}$ uniquely determine $\mathbf{h}$.

(Chruściel–Delay, 2011) Stationary Lorentzian settings.

- Answered correspondence question for stationary (t-independent) $\mathbf{g}$.
- Still an elliptic problem.

General Lorentzian/hyperbolic case: work in progress.

- (Joint with G. Holzegel).

Section 3

Unique Continuation for Waves

Waves on aAdS Spacetimes

Consider now a model problem:

- UC for Klein–Gordon equation from $\mathscr{I}$ ...
- ... on fixed aAdS spacetime.

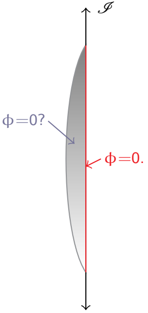
Question (Model Problem)

ϕ: solution on fixed aAdS spacetime $(\mathscr{M}, g)$ of

$$(\Box_g + \sigma)\phi = \mathcal{G}(\phi, \nabla\phi), \qquad \sigma \in \mathbb{R}.$$

Assume ϕ has zero Dirichlet and Neumann data on $\mathscr{I}$.

- *Is $\phi = 0$ locally near $\mathscr{I}$?*



The model problem.

Why the Wave Equation?

Crucial step toward main correspondence problem.

- EVE $\Rightarrow$ curvature, connection satisfy nonlinear wave equations.
- Shows main mechanism for uniqueness.

Model problem also has other applications:

- Rigidity results.
- Symmetry extension results.

Remark. The scalar mass σ is essential.

- σ determines asymptotics of ϕ near $\mathscr{I}$.

Some Intuition

Consider $(\Box_{g_0} + \sigma)\phi = 0$ on pure AdS:

- (Over)assume ϕ depends only on ρ.
- 2nd-order singular ODE $\Rightarrow$ two branches of solutions:

$$\phi_\pm = \rho^{\beta_\pm} \sum_{k=0}^{\infty} a_k^\pm \rho^k, \qquad \beta_\pm = \frac{n}{2} \pm \sqrt{\frac{n^2}{4} - \sigma}.$$

  - ϕ_-, ϕ_+: "Dirichlet and Neumann branches".
- For ϕ to vanish, must at least eliminate both branches:

$$\lim_{\rho \searrow 0} (\rho^{-\beta_+} \phi) \to 0.$$

Remark. Other ways to derive asymptotics ("FG", energy).

The Main Theorem I

Theorem (Holzegel–S., 2017)

Let $(\mathscr{M}, \mathbf{g})$ *be an aAdS spacetime.*

- *Assume* $\mathbf{g}$ *satisfies the "null convexity criterion".*

Let ϕ *be a* C^2*-solution of*

$$|(\Box_{\mathbf{g}} + \sigma)\phi| \leq \rho^{2+p}|\nabla_{t,\rho,\mathbb{S}^2}\phi| + \rho^p|\phi|, \qquad (\sigma \in \mathbb{R},\ p > 0).$$

- *Suppose* ϕ *satisfies vanishing condition as* $\rho \searrow 0$*:*

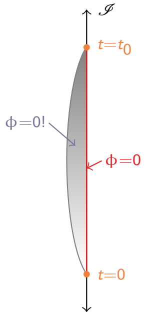
$$|\rho^{-\beta_+}\phi| + |\nabla_{t,\rho,\mathbb{S}^2}(\rho^{-\beta_+ + 1}\phi)| \to 0, \quad \text{if } \sigma \leq (n^2 - 1)/4,$$

$$|\rho^{-\frac{n+1}{2}}\phi| + |\nabla_{t,\rho,\mathbb{S}^2}(\rho^{-\frac{n-1}{2}}\phi)| \to 0, \quad \text{if } \sigma > (n^2 - 1)/4.$$

- *Vanishing condition holds on large enough time interval,*

$$0 \leq t \leq t_0.$$

Then, ϕ *vanishes in the interior, near* $\mathscr{I} \cap \{0 < t < t_0\}$*.*



Some Remarks

1. First such correspondence result in dynamical, non-analytic setting.
2. Vanishing condition: optimal when $\sigma \leq (n^2 - 1)/4$.
   - $\sigma = (n^2 - 1)/4$: conformal mass.
3. Result also holds for tensorial waves.

Below, we focus on:

- Null convexity criterion.
- Sufficiently large time interval.

Sufficiently Large Times

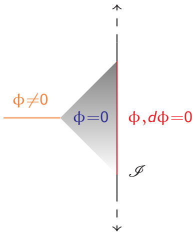
The sufficiently large time interval assumption is new.

- Needed for non-analytic wave equations.
- AdS: Need $t_0 > \pi$ (more than 1 AdS cycle).

Clearly needed for global UC results.

- Due to finite speed of propagation.

Also, seems necessary for local UC near $\mathscr{I}$.



Global UC fails.

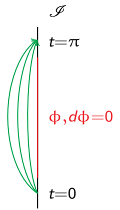
Short Time Intervals

Conjecture

On AdS, result is false if $t_0 < \pi$.

Special property of AdS geometry:

- $\exists$ family of future null geodesics from $\mathscr{I} \cap \{t = 0\}$...
- ... that are arbitrarily close to $\mathscr{I}$...
- ... and refocus at $\mathscr{I} \cap \{t = \pi\}$.



Null geodesics near $\mathscr{I}$.

Idea: Counterexamples via geometric optics or Gaussian beams.

- Construct solutions concentrated near these null geodesics.
- Similar to (Alinhac–Baouendi), (Ralston; Sbierski).

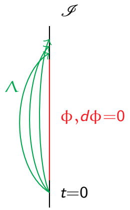
Null Geodesics

Extend ideas to aAdS spacetime $(\mathscr{M}, g)$.

- Consider (t-parametrized) null geodesics near $\mathscr{I}$:

$$\Lambda(t) = (\rho(t), \lambda(t)).$$

- λ: (coordinate) projection of Λ onto $\mathscr{I}$.
- $0 < \dot{\rho}(0) \ll 1$, and $\dot{\lambda}(0)$ almost $\mathring{\mathfrak{g}}$-null.



Null geodesics near $\mathscr{I}$.

Observation. Geodesic equation for ρ:

$$\ddot{\rho} - \frac{1}{2} \mathscr{L}_t \mathfrak{g}^{(0)}(\dot{\lambda}, \dot{\lambda}) \cdot \dot{\rho} - \mathfrak{g}^{(2)}(\dot{\lambda}, \dot{\lambda}) \cdot \rho + \text{l.o.t.} = 0.$$

- Leading-order behavior: damped harmonic oscillator.
- Geodesic motion driven by $\mathfrak{g}^{(2)}$ and $\mathscr{L}_t \mathfrak{g}^{(0)}$ (in null directions).

The Geodesic Return Theorem

Theorem (S, 2019)

Suppose the following conditions hold:

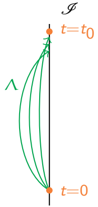
$$-\mathfrak{g}^{(2)}(X, X) \geq C > 0, \qquad |\mathcal{L}_t \mathfrak{g}^{(0)}(X, X)| \leq B \ll C.$$

- *X: any t-normalized* $\mathfrak{g}^{(0)}$*-null vector on* $\mathscr{I}$*.*

Then, there exists $t_0 = t_0(C, B) > 0$ *such that:*

- **If:** Λ *is a* g*-null geodesic starting from* $\mathscr{I}$ *at* $t = 0$*.*
- **If:** *Initial angle between* Λ *and* $\mathscr{I}$ *is* ε*-small.*
- **Then:** Λ *remains* ε*-close to* $\mathscr{I}$*.*
- **Then:** Λ *returns to* $\mathscr{I}$ *before* $t = t_0$*.*

Also: t_0 *is the same as in the main result!*



Null geodesics near $\mathscr{I}$.

Finding Pseudoconvexity

Key step in proof of main result.

- Find pseudoconvex foliation of hypersurfaces near $\mathscr{I}$.
- Prove Carleman estimates near $\mathscr{I}$ using this foliation.
- **Remark.** $\mathscr{I}$ fails to be pseudoconvex.

Lemma (Null Convexity Criterion)

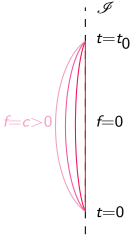
Suppose the following conditions hold:

$$-\mathfrak{g}^{(2)}(X,X) \geq C > 0, \qquad |\mathscr{L}_t \mathfrak{g}^{(0)}| \leq B \ll C.$$

- *X: t-normalized $\mathfrak{g}^{(0)}$-null vector.*

Then, there is a pseudoconvex foliation near $\mathscr{I}$...

- *... spanning a sufficiently long time interval $[0, t_0]$ on $\mathscr{I}$.*
- *$t_0 = t_0(C, B)$ same as before.*



Pseudoconvex hypersurfaces near $\mathscr{I}$ (in red).

The Main Theorem II

Theorem (Holzegel–S., 2017)

Let $(\mathscr{M}, \mathrm{g})$ *be an aAdS spacetime, satisfying the conditions*

$$-\mathfrak{g}^{(2)}(X, X) \geq C > 0, \qquad |\mathscr{L}_t \mathfrak{g}^{(0)}| \leq B \ll C.$$

Let ϕ *be a* C^2*-solution of*

$$|(\Box_{\mathrm{g}} + \sigma)\phi| \leq \rho^{2+p} |\nabla_{t,\rho,\mathbb{S}^2} \phi| + \rho^p |\phi|, \qquad (\sigma \in \mathbb{R},\ p > 0).$$

- *Suppose* ϕ *satisfies vanishing condition as* $\rho \searrow 0$*:*

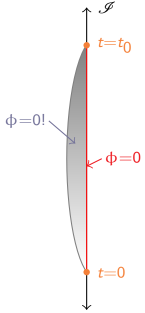
$$|\rho^{-\beta_+}\phi| + |\nabla_{t,\rho,\mathbb{S}^2}(\rho^{-\beta_+ + 1}\phi)| \to 0, \quad \text{if } \sigma \leq (n^2 - 1)/4,$$

$$|\rho^{-\frac{n+1}{2}}\phi| + |\nabla_{t,\rho,\mathbb{S}^2}(\rho^{-\frac{n-1}{2}}\phi)| \to 0, \quad \text{if } \sigma \geq (n^2 - 1)/4.$$

- *Vanishing condition holds on large enough time interval,*

$$0 \leq t \leq t_0 = t_0(C, B).$$

Then, ϕ *vanishes in the interior, near* $\mathscr{I} \cap \{0 < t < t_0\}$.



Einstein-Vacuum Spacetimes

Previous theorem needs not assume spacetime is vacuum.

If $(\mathscr{M}, g)$ also satisfies EVE:

- Null convexity criterion depends only on $\mathfrak{g}^{(0)} = \mathring{\mathfrak{g}}$.
- $(\mathscr{I}, \mathring{\mathfrak{g}})$ is static: null convexity criterion $\Leftrightarrow \mathcal{S}$ has positive Ricci curvature.

Example

- AdS, Schwarzschild-AdS, and Kerr-AdS satisfy the criterion.
- Planar/toroidal AdS $(\mathbb{R} \times \mathbb{S}^{n-1} \rightarrow \mathbb{R} \times \mathbb{R}^{n-1})$ fails the criterion.

Section 4

Applications

Rigidity of AdS

(Holzegel–S., 2015) Linearized EVE about AdS.

- Solution: divergence-free Weyl field $\mathbf{V}$.
- **Theorem:** If $\mathbf{V} \rightarrow 0$ fast enough at $\mathscr{I}$, for timespan $> \pi$, then $\mathbf{V} = 0$ inside.
- **Idea:** $\mathbf{V}$ satisfies tensorial wave equation.

Question (Rigidity of AdS)

Nonlinear version of the above.

- **Assume:** *Weyl curvature* $\mathbf{W} \rightarrow 0$ *fast enough at* $\mathscr{I}$, *for timespan* $> \pi$.
- **Goal:** $(\mathscr{M}, \mathbf{g})$ *must be AdS spacetime.*

Extension of Symmetries

Question (Inheritance of Symmetries)

If $(\mathscr{I}, \mathring{\mathfrak{g}})$ *has a symmetry, then is it also inherited by* $(\mathscr{M}, g)$*?*

Theorem (Holzegel–S., TBA)

Suppose $n \geq 3$*, and suppose* $(\mathscr{M}, g)$ *satisfies EVE and the null convexity criterion. If there is a vector field* Z *on* $\mathscr{I}$ *such that*

$$\mathscr{L}_Z \mathfrak{g}^{(0)} = \mathscr{L}_Z \mathring{\mathfrak{g}} = 0, \qquad \mathscr{L}_Z \mathfrak{g}^{(n)} = 0,$$

on large enough time interval, then Z *extends to a Killing vector field near* $\mathscr{I}$*.*

(Chruściel–Delay, 2011) Result for stationary spacetimes.

Rigidity of Kerr-AdS

Corollary (Rigidity of Schwarzschild-AdS)

If $\mathfrak{g}^{(0)}, \mathfrak{g}^{(n)}$ *are spherically symmetric, then:*

- $(\mathscr{M}, g)$ *must be Schwarzschild-AdS*...
- *...up to the photon sphere.*

Question (Rigidity of Kerr-AdS)

If $\mathfrak{g}^{(0)}, \mathfrak{g}^{(n)}$ *are axially symmetric, then must* $(\mathscr{M}, g)$ *be Kerr-AdS?*

Proof Outline

Key steps to proving symmetry extension:

1. Guess extension of Z into $\mathscr{M}$.
2. Find system of equations for which UC applies.
    - Similar to other UC results in relativity.
    - (Ionescu–Klainerman, Alexakis–Ionescu–Klainerman, Alexakis–Schlue)
3. Connect assumptions $\mathscr{L}_Z \mathfrak{g}^{(0)} = \mathscr{L}_Z \mathfrak{g}^{(n)} = 0$ to UC result.
    - From extended (partial) FG expansions.

Step 1. FG gauge makes this easy:

- Transport Z along ρ-coordinate.

The Wave-Transport System

Step 2. Derive closed system of:

- Wave equations for components ϕ of $\mathscr{L}_Z \mathbf{W}$:

$$(\Box_{\mathbf{g}} + c_\phi + \dots)\phi = \text{l.o.t.}(\phi, \nabla\phi, \psi, \nabla\psi).$$

- Transport equations for components ψ of $\mathscr{L}_Z \mathbf{g}$:

$$(\nabla_\rho + c_\psi)\psi = \text{l.o.t.}(\phi, \psi), \qquad (\nabla_\rho + c_\psi)\nabla\psi = \text{l.o.t.}(\phi, \nabla\phi, \psi, \nabla\psi).$$

Can derive coupled Carleman estimates for ϕ and ψ.

- If $(\phi, \psi) \rightarrow 0$ sufficiently fast at $\mathscr{I}$, for large enough time interval...
- ...then (ϕ, ψ) vanish near $\mathscr{I}$.

Derivation of Vanishing

Step 3. Derive vanishing conditions for (ϕ, ψ) at $\mathscr{I}$.

- FG expansions for (components of) $\mathfrak{g}$, $\mathbf{W}$:

$$\mathfrak{g} = \sum_{k=0}^{n-1} \rho^k \mathfrak{g}^{(k)} + \rho^n (\log \rho) \mathfrak{g}^{(*)} + \rho^n \mathfrak{g}^{(n)} + o(\rho^n),$$

$$\text{“}\mathbf{W}\text{”} = \sum_{k=0}^{n-1} \rho^k \mathfrak{w}^{(k)} + \rho^n (\log \rho) \mathfrak{w}^{(*)} + \rho^n \mathfrak{w}^{(n)} + o(\rho^n).$$

- $\mathfrak{g}^{(k)}$'s and $\mathfrak{w}^{(k)}$'s only depend on $\mathfrak{g}^{(0)}$ and $\mathfrak{g}^{(n)}$.
- $\Rightarrow \mathscr{L}_Z \mathfrak{g}^{(k)} = 0$ and $\mathscr{L}_Z \mathfrak{w}^{(k)} = 0$
- $\Rightarrow$ High-order vanishing for (ϕ, ψ).
- Equations from Step 2 $\Rightarrow$ ∞-order vanishing for (ϕ, ψ).

The Correspondence Problem

Question

Assuming EVE, do $\mathfrak{g}^{(0)}, \mathfrak{g}^{(n)}$ determine g (near $\mathscr{I}$)?

- *Work in progress (with G. Holzegel).*

Idea. $\mathbf{W}$ satisfies tensor wave equation.

- **Difficulty.** Two systems of wave equations, with two metrics.
- $\Rightarrow$ Terms with $g_1 - g_2$ and derivatives.

Challenge. Finding a closed system of PDEs.

- (Biquard) Found closed system in elliptic case.
- Hyperbolic case controls one less derivative.

Additional Questions

Question

Construction of counterexamples to UC?

- *In particular, for short time intervals.*

Question

Correspondence problems for Einstein + matter?

- *Einstein-scalar, Einstein-Maxwell, Einstein-Vlasov.*
- $\mathfrak{g}^{(k)}$*'s may also depend on matter field.*
- **Q.** *Can boundary data for matter yield better/worse results?*
- *Work in progress (Alex McGill).*