Unique Continuation for Waves, Carleman Estimates, and Applications

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Outline

1. Recent unique continuation (UC) results for wave equations.
    - UC “from infinity”.
2. Theory of UC.
    - Why is “classical” theory not enough?
3. Some ideas behind Carleman estimates.
    - Main analysis tool for UC.
4. (Other) Applications of Carleman estimates
    - Focus on control theory.

Section 1

Unique Continuation from Infinity

Wave Equations

Consider the wave equation:

$$\Box\phi := (-\partial_t^2 + \Delta_x)\phi = 0, \qquad \phi : \mathbb{R}_t \times \mathbb{R}_x^n \to \mathbb{R}.$$

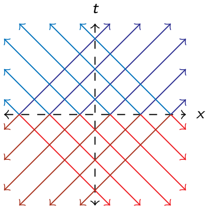
- Generalisations: linear/nonlinear waves, systems, geometric waves.
- Physics: Maxwell equations, Yang-Mills equations, Einstein equations, fluids

Initial value problem:

$$\Box\phi = F(t, x, \phi, D\phi), \qquad \phi|_{t=0} = \phi_0, \quad \partial_t\phi|_{t=0} = \phi_1.$$

- In general, $\exists!$ solution for "nice" initial data (ϕ_0, ϕ_1).
- Solution "depends continuously on" initial data.

Radiation



Propagation of waves ($\mathbb{R}^{1+1}$).

Regular solutions of $\Box\phi = 0$:

- Propagate at fixed, finite speed.
- Decay in space and time at known rates.

Can make sense of "asymptotics at infinity":

- Leading order coefficient: radiation field.

Question (UC from infinity)

Are solutions of wave equations determined by its "data at infinity"?

Minkowski Geometry

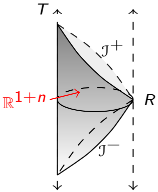
Theme: Geometric viewpoint for studying wave equations.

- Robust techniques applicable to many curved backgrounds.
- Applications to problems in relativity.

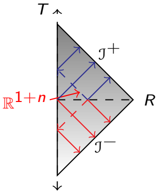
Natural setting: Minkowski spacetime $(\mathbb{R}^{1+n}, m)$.

- Minkowski metric: $m := -dt^2 + d(x^1)^2 + \cdots + d(x^n)^2$.
- Setting of special relativity.
- $\Box = m^{\alpha\beta} \nabla_{\alpha\beta}$: natural second-order PDO in Minkowski geometry.
  - Analogue of Δ in Euclidean geometry.

Infinity



$\mathbb{R} \times \mathbb{S}^n$, mod $\mathbb{S}^{n-1}$.



Previous picture, projected.

Infinity visualised via Penrose compactification.

- Conformal transformation $m \mapsto \Omega^2 m$.
- $(\mathbb{R}^{1+n}, \Omega^2 m)$ isometrically embeds into relatively compact region in $\mathbb{R} \times \mathbb{S}^n$.

Infinity realised as boundary of shaded region.

- Future/past null infinity $\mathcal{I}^\pm$: Null geodesics (bicharacteristics of $\Box$) terminate here.
- Radiation field manifested at $\mathcal{I}^\pm$.

For this talk, useful for drawing pictures.

Main Questions

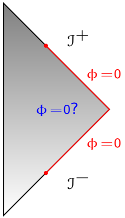
Question (UC from infinity)

Does ϕ on some part of $\mathcal{I}^\pm$ determine ϕ inside?

- *General linear/nonlinear waves, e.g.,* $(\Box + \nabla_X + V)\phi = 0$*?*
- *Geometric waves on curved backgrounds:* $\Box_g \phi = g^{\alpha\beta} \nabla^2_{\alpha\beta} \phi = \dots$*?*

For linear waves:

- If $\phi = 0$ on some part of $\mathcal{I}^\pm$, then is $\phi = 0$ inside?
- Are nonradiating waves trivial?



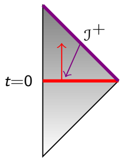
Scattering Results

(Friedlander) Isometry between initial data (at $t = 0$) and radiation field (at $\mathcal{I}^+$).

- Applies only to free waves ($\Box\phi = 0$).

Various generalisations:

- Product manifolds $\mathbb{R} \times X$, special nonlinear waves, special black hole spacetimes.



Red: Solve forward from $t = 0$.

Blue: Solve backward from $\mathcal{I}^+$.

However, we are more interested in:

- Ill-posed settings: cannot solve the wave equation.
- Other linear and geometric waves.

Result Near Infinity

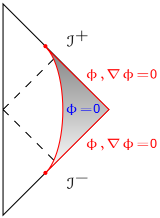
Theorem (Alexakis–Schlue–S., 2015)

Assume ϕ is a solution, near $\mathcal{I}^{\pm}$, of

$$\Box\phi + \nabla_X\phi + V\phi = 0,$$

where X, V decay sufficiently toward $\mathcal{I}^{\pm}$.

If ϕ, $\nabla\phi$ vanish to ∞-order on $(\frac{1}{2}+\varepsilon)\mathcal{I}^{\pm}$, then $\phi = 0$ in the interior near $(\frac{1}{2}+\varepsilon)\mathcal{I}^{\pm}$.



Remark. The ∞-order vanishing is optimal.

- Counterexamples if ϕ vanishes only to finite order.
- On $\mathbb{R}^{1+n}$ $(n > 2)$, can take $\phi = \nabla_x^k r^{-(n-2)}$.

Geometric Robustness

Question

Can UC result be extended to curved backgrounds?

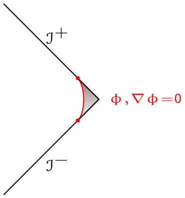
- *Asymptotically flat spacetimes: those with "similar structure of infinity".*

Theorem (Alexakis–Schlue–S., 2015)

The main result extends to a large class of (both stationary and dynamic) asymptotically flat spacetimes, including:

1. *Perturbations of Minkowski spacetimes.*
2. *Schwarzschild and Kerr spacetimes, and perturbations.*

For (2), result can be localised near $\varepsilon\mathcal{I}^{\pm}$.



Finite-Order Vanishing?

Question

On Minkowski spacetime $(\mathbb{R}^{1+n}, m)$*:*

- *Can* ∞*-order vanishing condition be somehow removed?*

Recall. Counterexamples $\nabla_x^k r^{-(n-2)}$.

- Note that these blow up at $r = 0$.

Idea. Impose global regularity for ϕ, up to $r = 0$.

The Global Result

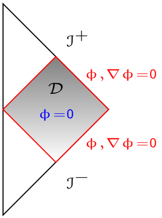
Theorem (Alexakis–S., 2015)

Assume ϕ is a regular solution in $\mathcal{D}$ of

$$\Box\phi + V\phi = 0, \qquad \|V\|_{L^\infty} \leq C,$$

where V also decays toward $\mathcal{I}^\pm$ as before.

If ϕ, $\nabla\phi$ vanish to order $A > A_0$ at $\frac{1}{2}\mathcal{I}^\pm$, with A_0 depending on C, then $\phi = 0$ everywhere on $\mathcal{D}$ (and hence $\mathbb{R}^{1+n}$).



Remark. The L^∞-assumption on V is necessary.

- Otherwise, there are counterexamples.

The Global Nonlinear Theorem

Question

Can some special wave equations be better behaved?

Theorem (Alexakis–S., 2015)

Suppose ϕ is a regular solution in $\mathcal{D}$ of

$$\Box\phi + V|\phi|^{p-1}\phi = 0, \qquad p \geq 1,$$

where V satisfies a monotonicity property (depending on p).

If ϕ, $\nabla\phi$ vanishes to order δ at $\frac{1}{2}\mathcal{I}^{\pm}$ for any $\delta > 0$, then $\phi = 0$ on $\mathcal{D}$.

Idea. Estimates not for $\Box$, but for

$$\Box_{V,p}\phi := \Box\phi + V \cdot |\phi|^{p-1}\phi.$$

Section 2

Unique Continuation Theory

Unique Continuation

Unique continuation (UC): classical problem in PDEs.

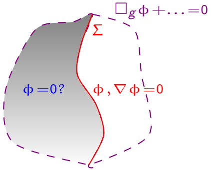
- When we cannot solve a PDE, we can still ask if solutions are unique.

Problem (Unique Continuation)

Suppose:

- ϕ *solves* $(\Box_g + \nabla_X + V)\phi = 0$.
- $\phi, \nabla\phi$ *vanish on a hypersurface* Σ.

Must ϕ *vanish on one side of* Σ*?*



In particular, we are interested in $\Sigma \subseteq \mathcal{I}^{\pm}$.

The Classical Theory

Ancient theory: analytic PDE, noncharacteristic Σ.

- (Cauchy–Kovalevskaya) Existence, uniqueness of analytic solutions.
- (Holmgren, F. John) Solution unique even in nonanalytic classes.

Classical theory for non-analytic equations (Calderón, Hörmander):

- Crucial point: pseudoconvexity of Σ.
- Σ pseudoconvex $\Rightarrow$ Carleman estimates $\Rightarrow$ UC from Σ.
- (Alinhac–Baouendi) Σ not pseudoconvex $\Rightarrow \exists\, X,\ V$ with counterexamples.

Remark. Classical UC results are purely local.

- UC from a small neighbourhood of $P \in \Sigma$.

A Geometric Perspective

(Lerner–Robbiano) The following are equivalent:

- $\Sigma := \{f = 0\}$ is pseudoconvex (wrt $\Box_g$ and f).
- $\nabla^2 f(X, X) < 0$ on Σ, whenever $g(X, X) = Xf = 0$.
- $-f$ is convex on Σ, in the tangent null (bicharacteristic) directions.

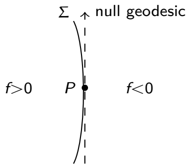
In this case, UC from Σ to $f > 0$.

Visual interpretation:

- Null geodesic (bicharacteristic) hitting Σ tangentially...
- ... lies in $\{f < 0\}$ nearby.

Note. Pseudoconvexity is conformally invariant.

- Sensible to take $\Sigma \subseteq \mathcal{I}^{\pm}$.



Σ pseudoconvex at *P*.

Zero Pseudoconvexity

Bad news. $\mathcal{I}^{\pm}$ (barely) fails to be pseudoconvex.

- "Zero pseudoconvex".

Σ is zero pseudoconvex $\Leftrightarrow$ Σ is ruled by null geodesics.

- Need more refined understanding of geometry near $\mathcal{I}^{\pm}$.

Possible loss of local UC in zero pseudoconvex settings:

- (Alinhac–Baouendi) Counterexample to local UC when $\Sigma = \{x_n = 0\} \subseteq \mathbb{R}^{1+n}$.
- (Kenig–Ruiz–Sogge) Global UC from *all* of $\Sigma = \{x_n = 0\}$.
- Main result: Semi-global UC (from "large enough" hypersurface $(\frac{1}{2} + \varepsilon)\mathcal{I}^{\pm}$).
- Main result: Local UC (locally from $\varepsilon\mathcal{I}^{\pm}$).

Carleman Estimates

Carleman estimates: main technical tool for proving UC.

- Weighted integral estimates, with free parameter $\lambda > 0$.
- (Carleman, Calderón, Hörmander, Tataru, ...)

$$\lambda \int_\Omega w_\lambda (|\nabla \phi|^2 + |\phi|^2)] \lesssim \int_\Omega w_\lambda |\Box_g \phi|^2 .$$

- Ω: spacetime region.
- w_λ: weight function (constructed from pseudoconvexity).

Σ pseudoconvex $\Rightarrow$ (local) Carleman estimate near Σ.

- **Q.** Zero pseudoconvex $\Rightarrow$ "degenerate" Carleman estimates?

Section 3

Carleman Estimates: Some Key Ideas

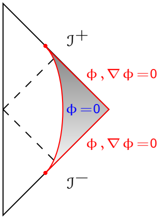
Result Near $\mathcal{I}^\pm$: Pseudoconvexity

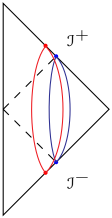
Recall. UC result on Minkowski:

- UC from $(\frac{1}{2} + \varepsilon)\mathcal{I}^\pm$.

Consider hyperboloids in $\mathbb{R}^{1+n}$:

- Blue: level sets of $f = r^2 - t^2$.
  - These are only zero pseudoconvex.
- Red: "warped" level sets of $f_\star$.
  - These are (inward) pseudoconvex.
  - Pseudoconvexity degenerates at $\mathcal{I}^\pm$.

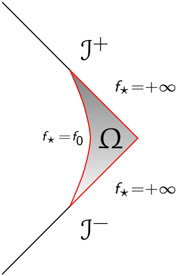




Result Near $\mathscr{I}^{\pm}$: Infinite-Order Vanishing

Can derive Carleman estimate roughly of the form:

$$\int_{\Omega} f_{\star}^{2\lambda} (w|\nabla\phi|^2 + \phi^2) \lesssim \lambda^{-1} \int_{\Omega} f_{\star}^{2\lambda} |\Box\phi|^2 + \int_{f_{\star}=\infty} f_{\star}^{2\lambda} (|\nabla\phi|^2 + \phi^2).$$



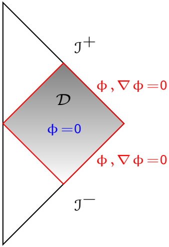
Need boundary term at $f_\star = +\infty$ to vanish:

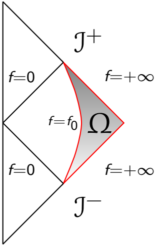
- Need 4λ-order vanishing for ϕ, $\nabla\phi$.

Must assume $\phi, \nabla\phi$ vanish at $f_\star = f_0$.

- In practice, done using cutoff function.
- $\Rightarrow$ For UC, need to take $\lambda \nearrow \infty$.
- $\Rightarrow$ Need ∞-order vanishing at $\mathscr{I}^{\pm}$.

Global Result: Carleman Near $\mathcal{I}^\pm$





Recall. Global UC result on Minkowski:

- Global UC from $\frac{1}{2}\mathcal{I}^\pm$, with finite-order vanishing.

Carleman estimate in this setting: (roughly)

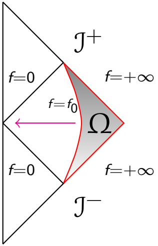
$$\int_\Omega f^{2\lambda}\phi^2 \lesssim \lambda^{-1}\int_\Omega f^{2\lambda}|\Box\phi|^2 + \int_{f=\infty} f^{2\lambda}(|\nabla\phi|^2 + \phi^2).$$

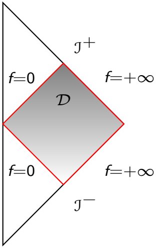
- $f := r^2 - t^2$.
- **Remark.** Also need lower-order modification of $f^{2\lambda}$.

Remark. No $|\nabla\phi|^2$-term on LHS:

- Since level sets of f are zero pseudoconvex.
- Can only handle equations of the form $(\Box + V)\phi = 0$.

Global Result: Global Carleman





To avoid ∞-order vanishing:

- $\Rightarrow$ Avoid taking $\lambda \nearrow \infty$.
- $\Rightarrow$ Avoid using cutoff function near $f = f_0$.

Idea. Note weight $f^{2\lambda}$ vanishes on $f = 0$.

- $f = 0$: null cone about origin ($|t| = r$).
- If Carleman estimate can be pushed to $f = 0$, then we do not need a cutoff to kill the $f = f_0$ boundary term.

Can derive global Carleman estimate: (roughly)

$$\int_{\mathcal{D}} f^{2\lambda}\phi^2 \lesssim \lambda^{-1} \int_{\mathcal{D}} f^{2\lambda}|\Box\phi|^2 + \int_{f=\infty} f^{2\lambda}(|\nabla\phi|^2 + \phi^2).$$

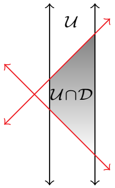
Global Carleman: Finite Domains

Idea. Estimate holds on finite spacetime domains:

- Given $\mathcal{U} \subseteq \mathbb{R}^{1+n}$:

$$\int_{\mathcal{U}\cap\mathcal{D}} f^{2\lambda} \phi^2 \lesssim \lambda^{-1} \int_{\mathcal{U}\cap\mathcal{D}} f^{2\lambda} |\Box\phi|^2 + \int_{\partial\mathcal{U}\cap\mathcal{D}} (\dots).$$

- Extra boundary term on $\partial\mathcal{U}$.
- Novel feature: No boundary term anywhere on $\mathcal{D}$.



The finite setting allows for one more trick:

- Another modification of weight $f \Rightarrow$ can reinsert $|\nabla\phi|^2$ in LHS:

$$\int_{\mathcal{U}\cap\mathcal{D}} f_\dagger^{2\lambda} (w|\nabla\phi|^2 + \phi^2) \lesssim \lambda^{-1} \int_{\mathcal{U}\cap\mathcal{D}} f_\dagger^{2\lambda} |\Box\phi|^2 + \int_{\partial\mathcal{U}\cap\mathcal{D}} (\dots).$$

- Control of ϕ on "shaded bulk region" by ϕ on "black boundary".

Section 4

Applications of Carleman Estimates

Sample of Applications

Geometric UC results have applications to relativity:

- (Alexakis–Schlue) Nonexistence of time-periodic vacuum spacetimes.

Singularity formation for NLW (subconformal focusing):

- Finite Carleman estimate $\Rightarrow$ information about behaviour of singularities.

Control theory ($*$): Exact controllability of wave equations.

Inverse problems: Determining PDE from measurements of its solutions.

- Lower-order coefficients X, V.
- Metric (principal coefficients) g.

Exact Controllability

Recall. $\Omega \subseteq \mathbb{R}^n$: open, bounded, smooth boundary.

The following initial-boundary value problem has a unique solution:

- Wave equation: $\mathcal{L}\phi = (\Box + \nabla_X + V)\phi = 0$ on $[T_-, T_+] \times \Omega$.
- Initial condition: $(\phi, \partial_t \phi)|_{t=T_-} = (\phi_0^-, \phi_1^-)$.
- Boundary condition: $\phi|_{(T_-,T_+)\times\partial\Omega} = \phi_b$.

Problem (Exact Dirichlet boundary controllability)

Fix $\Gamma \subseteq (T_-, T_+) \times \partial\Omega$.

- *Given any "initial" and "final" data $(\phi_0^\pm, \phi_1^\pm) \in L^2(\Omega) \times H^{-1}(\Omega)$...*
- *... can one find Dirichlet boundary data $\phi_b \in L^2(\Gamma)$, such that...*
- *... the solution of the above satisfies $(\phi, \partial_t \phi)|_{t=T_+} = (\phi_0^+, \phi_1^+)$?*

In other words, can solutions be controlled via Dirichlet boundary data?

Basic Principles

Finite speed of propagation $\Rightarrow$ lower-bound on timespan $T_+ - T_-$.

- Information from ϕ_b needs time to travel to all of Ω.

(Dolecki–Russell, Lions) Hilbert uniqueness method (HUM)

Preceding problem can be solved if and only if:

- For any solution ψ satisfying

$$\mathcal{L}^* \psi|_{[T_-, T_+] \times \Omega} = 0, \quad (\psi, \partial_t \psi)|_{t = T_+} = (\psi_0^+, \psi_1^+), \quad \psi|_{(T_-, T_+) \times \partial \Omega} = 0 \ldots$$

- ...the following observability inequality holds:

$$\| (\psi_0^+, \psi_1^+) \|_{H^1(\Omega) \times L^2(\Omega)} \leq C \| \partial_\nu \psi \|_{L^2(\Gamma)} .$$

- $\partial_\nu \psi$: Neumann data.
- C independent of ψ.

Methods for Observability

Thus, exact controllability is reduced to proving observability.

I. Fourier series methods: handles $(-\partial_t^2 + \partial_x^2 + \alpha)\phi = 0$.

II. Multiplier (energy) methods: handles $\Box\phi = 0$.

- And some perturbations.

III. Microlocal methods:

- Most precise, optimal (w.r.t. control region) results.
- (Bardos–Lebeau–Rauch) Geometric control condition.
- However, only applies to time-independent (or time-analytic) equations.

Carleman Estimate Methods

IV. Carleman estimates: very robust method for observability.

- Handles time-dependent equations, without assuming analyticity.

Via multiplier/Carleman methods, can show:

- Observability estimate $\|(\psi_0^+, \psi_1^+)\|_{H^1(\Omega)\times L^2(\Omega)} \leq C\|\partial_\nu \psi\|_{L^2(\Gamma)} \dots$
- ...with $\Gamma := (T_-, T_+) \times \{x \in \partial\Omega \mid (x - x_0) \cdot \nu > 0\}$.
- $x_0 \in \mathbb{R}^n$ fixed.
- ν: outer unit normal to Ω.

Geometric interpretation:

- $(t, x) \in \Gamma \Leftrightarrow$ ray from x_0 through x is leaving Ω at x.

Novel Improvements I

Previous Carleman estimates + energy estimates $\Rightarrow$ observability.

- With some novel features.

A. Region Γ of control can be improved.

- Γ can be time-dependent:

$$\Gamma = [(T_-, T_+) \times \{x \in \partial\Omega \mid (x - x_0) \cdot \nu > 0\}] \cap \mathcal{D}_{(t_0, x_0)} .$$

- $\mathcal{D}$: exterior of null cone about (t_0, x_0).

Theorem (S.)

Exact controllability for general wave equations...

- *...with Dirichlet control on the above Γ, restricted to $\mathcal{D}_{(t_0, x_0)}$.*

Novel Improvements II

What about time-dependent domains with moving boundaries?

$$\mathcal{U} = \bigcup_{T_- \leq \tau \leq T_+} (\{\tau\} \times \Omega_\tau).$$

B. Carleman estimate proved using Lorentzian-geometric methods.

- Directly applicable to more general domains $\mathcal{U}$.
- $(x - x_0) \cdot \nu > 0$ replaced by similar condition, with a "relativistic correction".

Theorem (S.)

Previous theorem extends to time-dependent domains $\mathcal{U}$:

- Γ *similar to before, but with "relativistic correction".*
- *Achieves optimal timespan when* $n = 1$.

Some Final Context

Previous literature for time-dependent domains is sparse:

- General n: only special cases of $\mathcal{U}$.
  - $\mathcal{U}$ expanding (Bardos–Chen).
  - Self-similar and asymptotically cylindrical (Miranda).
- $n = 1$: recent work by various authors.
  - Optimal results for special cases ($\partial\mathcal{U} =$ two lines).
  - General cases: non-optimal timespan.

Future work. Explore controllability for geometric wave equations.

- Lorentzian-geometric techniques well-adapted to this analysis.
- General Lorentzian settings unexplored.

The End

Thank you for your attention!