THE NEED FOR REAL NUMBERS

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1. Some Things Are Just Not Rational

It was known even in ancient times that the *rational numbers*, i.e., those numbers that can be expressed as fractions of integers, could not suffice to describe all numerical quantities of interest. For example, the ancient Greeks had determined that square roots can be *irrational*, that is, not a rational number. Other well-known numbers can also be shown to be irrational, with standard examples including π and e. Here, we demonstrate some relatively simple proofs of these facts.

1.1. $\sqrt{2}$ is Irrational.

Consider a right isosceles triangle, such that the sides bordering the right angle have length 1. By the Pythagorean theorem, then the remaining side - the hypotenuse - has length h, with

$$h^2 = 1^2 + 1^2 = 2.$$

In other words, h itself is the square root of 2. Thus, this quantity $\sqrt{2}$ arises naturally from constructing geometric quantities with unit lengths. Below, we give a simple proof by contradiction that $\sqrt{2}$ is indeed irrational. [1]

Suppose (for an eventual contradiction) that $\sqrt{2}$ is rational, of the form

$$\sqrt{2} = \frac{a}{b},$$

where a, b are positive integers and have no common factors. Squaring this yields

$$2 = \frac{a^2}{b^2}, \qquad a^2 = 2b^2.$$

From this, we see that a^2 must be even. This implies that a itself is even (since if a is odd, then so is a^2).

Since a is even, then $a = 2y$ for some integer y. Returning to our rationality assumption for $\sqrt{2}$ and doing a bit of algebra, then

$$2 = \frac{a^2}{b^2} = \frac{4y^2}{b^2}, \qquad b^2 = \frac{4y^2}{2} = 2y^2.$$

This now implies that b^2 is even, so that b is also even.

We have now shown that a and b are both even. But, we had assumed that a and b had no common factors! Thus, from our assumption that $\sqrt{2}$ is rational, we have arrived at a contradiction. This shows that $\sqrt{2}$ is not rational.

Exercise 1. *Can you tweak the above proof to also show that:*

(1) $\sqrt{p}$ *is irrational for any prime number* p*?*
(2) $\sqrt{n}$ *is irrational for any natural number* n *that is not a perfect square?*
(3) *What about the* k*-th root of any natural number* n*?*

[1] However, this is by no means the only proof of this fact.

1.2. π **is Irrational.** Next, using basic calculus, we give a proof (out of many available) of the fact that π, the ratio between the circumference and the diameter of a circle, is irrational. This argument uses the following ingredients:

- Elementary single-variable calculus: derivatives, integrals, etc.
- The (precalculus) fact that π is the first positive zero of $\theta \mapsto \sin \theta$.

Suppose (again for a contradiction) that π is rational, with

$$\pi = \frac{a}{b},$$

where a and b are once again positive integers without common factors. For any natural number n, we construct the following function on the interval $[0, \pi]$:

$$f_n : [0, \pi] \to \mathbb{R}, \qquad f_n(x) = \frac{b^n x^n (\pi - x)^n}{n!} = \frac{x^n (a - bx)^n}{n!}.$$

In order to obtain our contradiction, we consider the integrals

$$I_n = \int_0^\pi f_n(x) \sin x dx, \qquad n \in \mathbb{N}.$$

First, since the integrand is strictly positive on $(0, \infty)$, then every I_n is strictly positive. The other preliminary observation is that the f_n's become very small as n becomes large. This is described quantitatively through the following claim:

Claim 1. *The maximum of $f_n(x)$ occurs at $x = \pi/2$, and*

$$f_n(\pi/2) = \frac{b^n \pi^{2n}}{2^{2n} n!}.$$

Moreover, these maxima converge to 0 *as $n \nearrow \infty$, that is*

$$\lim_{n \nearrow \infty} f_n(\pi/2) = 0.$$

Since each f_n is strictly positive on $(0, \pi)$, the claim implies that the f_n's become smaller and smaller and decrease to zero as $n \nearrow \infty$. Since the f_n's become uniformly arbitrarily small as n becomes large, it follows that the I_n's must become arbitrarily small as well. Thus, for sufficiently large n, we must have $0 < I_n < 1$.

With the above observations in hand, we can now derive a contradiction:

Claim 2. *I_n is an integer for any n.*

This claim is a consequence of our (erroneous) assumption that π is rational. Indeed, this claim contradicts our previous observation that $0 < I_n < 1$! The above outlines the entire proof process. In order to complete this proof and to show that π is irrational, we need only prove the above two claims.

Proof of Claim 1. Using the product rule, we can compute

$$f_n'(x) = \frac{b^n}{(n-1)!} [x^{n-1} (\pi - x)^n - x^n (\pi - x)^{n-1}].$$

Therefore, $f_n'(x)$ vanishes whenever

$$x^{n-1} (\pi - x)^n = x^n (\pi - x)^{n-1},$$

that is, whenever one of the following holds:

$$x = 0 \text{ (if } n > 1), \qquad x = \pi \text{ (if } n > 1), \qquad x = \pi - x.$$

The last case is equivalent to x being equal to $\pi/2$. Since

$$f_n(0) = f_n(\pi) = 0, \qquad f_n(\pi/2) = \frac{b^n \pi^{2n}}{2^{2n} n!},$$

then the maximum of f_n must occur at $x = \pi/2$, with the desired maximum value.

Finally, to see that

$$\lim_{n \nearrow \infty} \frac{b^n \pi^{2n}}{2^{2n} n!} = 0,$$

one needs only note that $n!$ grows strictly faster than $(b\pi^2/2^2)^n$. □

Proof of Claim 2. First, we compute the endpoints of all derivatives of f_n.

Claim 3. *The quantities $f_n^{(m)}(0)$ and $f_n^{(m)}(\pi)$ are:*
- 0*, if $m < n$ and $m > 2n$.*
- *Some integer, if $n \leq m \leq 2n$.*

Furthermore, $f_n^{(2n)}$ is an integer constant function.

One can prove Claim 3 directly using a bit of brute force. We defer this until after the remainder of the proof of Claim 2 is completed.

Now that we know the endpoints $f_n^{(m)}(0)$ and $f_n^{(m)}(\pi)$, the computation for I_n proceeds via repeated integrations by parts. First of all, we have

$$\begin{aligned}\int_0^\pi f_n^{(m)}(x) \sin x dx &= -\int_0^\pi f_n^{(m)}(x) \frac{d}{dx} \cos x dx \\ &= f_n^{(m)}(\pi) \cos \pi - f_n^{(m)}(0) \cos 0 + \int_0^\pi f_n^{(m+1)}(x) \cos x dx \\ &= \text{ (integer) } + \int_0^\pi f_n^{(m+1)}(x) \cos x dx,\end{aligned}$$

where we have applied Claim 3. By a similar computation, we also have

$$\int_0^\pi f_n^{(m)}(x) \cos x dx = \int_0^\pi f_n^{(m)}(x) \frac{d}{dx} \sin x dx = -\int_0^\pi f_n^{(m+1)}(x) \sin x dx.$$

Consequently, we can iterate as follows:

$$\begin{aligned} I_n &= \text{ (integer) } + \int_0^\pi f_n'(x) \cos x dx \\ &= \text{ (integer) } - \int_0^\pi f_n''(x) \sin x dx \\ &= \dots \\ &= \text{ (integer) } + (-1)^n \int_0^\pi f_n^{(2n)}(x) \sin x dx, \end{aligned}$$

Since $f_n^{(2n)}$ is an integer constant (by Claim 3), then the last integral on the right-hand side must be an integer (since both sin and cos can be integrated directly). This proves that I_n is indeed an integer. □

Proof of Claim 3. If $m < n$, then using the product and power rules, we see that $f_n^{(m)}(x)$ is a sum of terms of the form

$$\frac{Cb^n}{\min(p, q)!} x^p (\pi - x)^q,$$

where C is an integer constant, and where p and q are positive integers such that $p + q + m = 2n$. Since the above expression vanishes at $x = 0$ and $x = \pi$, then

$$f_n^{(m)}(0) = f_n^{(m)}(\pi) = 0.$$

Similarly, if $n \leq m \leq 2n$, then $f_n^{(m)}(x)$ is still a sum of terms of the form

$$g(x) = \frac{Cb^n}{\min(p, q)!} x^p (\pi - x)^q,$$

where C is an integer constant and $p + q + m = 2n$, except that now p or q can be zero. Again, it suffices to show that the above expression $g(x)$ is an integer when $x = 0$ or $x = \pi$. Suppose first that $x = 0$.

- If $p > 0$, then $g(x)$ vanishes.
- If $p = 0$, then $g(x) = Cb^n \pi^q = Ca^n b^{n-q}$, which is an integer.

Similarly, in the case $x = \pi$:

- If $q > 0$, then $g(x)$ vanishes.
- If $q = 0$, then $g(x) = Cb^n \pi^p = Ca^n b^{n-p}$, which is an integer.

It now follows that $f_n^{(m)}(0)$ and $f_n^{(m)}(\pi)$ are integers, when $n \leq m \leq 2n$.

Finally, by the power rule, taking $2n$ derivatives of f_n exhausts all the powers of both x and $(\pi - x)$, so that $f_n^{(2n)}$ must be an integer constant function. □

2. Filling In the Holes

In the preceding section, we demonstrated the necessity of going beyond the rational numbers by showing the existence of quantities (e.g., the hypotenuse of a right isosceles triangle and the circumference of a circle) which cannot be expressed merely as a fraction of integers. Now, we continue this discussion by describing some analytic reasons why one must consider not only the rationals. In fact, this extension from the rational numbers to the "real numbers" is fundamental to mathematical analysis, in particular to developing calculus.

Here, we examine two basic properties of real numbers – the "least upper bound property" and "completeness" – and we briefly explore how one can rigorously construct the real number line so that these properties are satisfied. The principal idea behind both properties is that the set of rational numbers contains "holes", or deficiencies, and that by extending to the real numbers, one "fills in these holes".

Note that these "holes" within the rationals must be infinitesimally small, since for any $p, q \in \mathbb{Q}$ with $p < q$, then there is some $r \in \mathbb{Q}$ such that $p < r < q$ (given any two distinct rational numbers, there exists a third rational number that is between those two). On the other hand, the set of irrational numbers must be strictly larger than the set of rational numbers. [2] Thus, from the point of view of set cardinality, these "holes" between the rationals must be relatively large.

2.1. **The Least Upper Bound Property.**

A most fundamental property of the real number line is the following, called the *least upper bound property*:

- *Every nonempty subset of* $\mathbb{R}$ *bounded from above has a least upper bound.*

To be more precise, if A is a subset of $\mathbb{R}$ that has an upper bound (i.e., an element $M \in \mathbb{R}$ such that $x \leq M$ for every $x \in A$), then there exists a least upper bound of A, that is, a number $\alpha \in \mathbb{R}$ such that:

[2] If the irrationals $\mathbb{R} \setminus \mathbb{Q}$ is countable, then so is $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$, which is a contradiction.

- α is an upper bound for A, in the sense mentioned above.
- α is less than or equal to any other upper bound for A.

This least upper bound of A is typically denoted $\sup A$. [3]

Exercise 2. *Can a subset $A \subseteq \mathbb{R}$ that is bounded from above have more than one least upper bound? Why or why not?*

Exercise 3. *Show that any $A \subseteq \mathbb{R}$ which is bounded from below has a greatest lower bound, which is typically denoted* $\inf A$. [4]

Note in particular that the set $\mathbb{Q}$ of rational numbers does not have the least upper bound property. Indeed, the subset

$$\{q \in \mathbb{Q} \mid q^2 < 2\},$$

that is, the set of all rational numbers with absolute value less than $\sqrt{2}$, is bounded from above but has no least upper bound in $\mathbb{Q}$.

The least upper bound property for $\mathbb{R}$ is essential for establishing many of the most basic properties of real numbers. Here, we focus on one such property which provides a clear contrast to a result proved in the previous section.

Theorem 2.1. *If x is a positive real number, and if $n \in \mathbb{N}$, then there exists a unique positive real number y such that $y^n = x$. In other words, every positive real number has a unique positive real n-th root.* [5]

Proof. [6] Consider the set

$$E = \{t \in \mathbb{R} \mid t^n < x\}.$$

This E is nonempty, since if $t = x/(1 + x)$, then

$$0 < t < 1, \qquad t^n < t < x,$$

so that $t \in E$. Moreover, E is bounded from above, since if $t > 1 + x$, then

$$t^n \geq t > x,$$

and hence $1 + x$ is an upper bound of E. Thus, by the least upper bound property, then E has a least upper bound $y = \sup E$.

We wish to show that y is in fact the n-th root of x. To accomplish this, we show that contradictions occur when one assumes either $y^n < x$ or $y^n > x$.

First, suppose $y^n < x$. Then, given $0 < h < 1$, we have

$$(y + h)^n - y^n = h \sum_{k=0}^{n-1} (y + h)^k y^{n-1-k} < hn(y + 1)^{n-1}.$$

If this h is sufficiently small (one can directly solve for how small h must be), then

$$(y + h)^n - y^n < hn(y + 1)^{n-1} < x - y^n, \qquad (y + h)^n < x.$$

Thus, $y + h \in E$. This contradicts that y is an upper bound for E.

[3] $\sup A$ is often referred to as the *supremum* of A.

[4] $\inf A$ is often referred to as the *infimum* of A.

[5] Here, we assume multiplication of real numbers is well-defined. The actual construction of the real numbers and its basic arithmetic operations, though, is rather nontrivial.

[6] From [1].

Next, suppose $y^n > x$. Then, for $0 < h < y$, we have

$$y^n - (y-h)^n = h \sum_{k=0}^{n-1} y^k (y-h)^{n-1-k} < hny^{n-1} < y^n - x,$$

as long as h is sufficiently small. Thus, with this h, we have $(y-h)^n > x$, which contradicts that y is the least upper bound for E. □

Due to the least upper bound property, the real numbers succeed where the rationals fail – one cannot take n-th roots of positive rational numbers while remaining within the rationals, but this can be done instead with the real numbers.

Exercise 4. *Prove the following* Archimedean property *for the real numbers: if x and y are positive real numbers, then there exists some $n \in \mathbb{N}$ such that $nx > y$. Hint: Assume the above is false, and apply the least upper bound property.*

The above demonstrates that the real numbers have many "nicer" properties than the rationals, thanks to the least upper bound property. But why, though, does $\mathbb{R}$ satisfy this least upper bound property? The answer lies in how the real number line is formally constructed. In other words, one must define the real number line rather carefully so that this property is indeed satisfied.

One common method for defining $\mathbb{R}$ is via *Dedekind cuts*, named after Richard Dedekind. A subset $A \subseteq \mathbb{Q}$ is called a *cut* iff the following conditions hold:

- $A \neq \emptyset$, and $A \neq \mathbb{Q}$.
- If $p \in A$, then A contains every rational number smaller than p.
- If $p \in A$, then there exists some $q \in A$ such that $p < q$. In other words, A contains no largest element.

The real number line $\mathbb{R}$ can then be defined as the set of all cuts.

The intuition behind this construction is the following: every "real number x" is defined to the set of all rational numbers strictly smaller than x. Note $\mathbb{R}$ can be thought of as containing the rational numbers, since each $p \in \mathbb{Q}$ can be identified with the cut $\{q \in \mathbb{Q} \mid q < p\}$ of all rationals smaller than p. However, $\mathbb{R}$ contains more than just $\mathbb{Q}$. For example, the "irrational number $\sqrt{2}$" is defined by the cut

$$\{q \in \mathbb{Q} \mid q < 0 \text{ or } q^2 < 2\}.$$

One advantage of the Dedekind cut method is that comparing real numbers is easy to define. Given cuts $x, y \in \mathbb{R}$, we can define $x \leq y$ to simply mean $x \subseteq y$. Indeed, if we think of x as "all the rationals less than x", then $x < y$ must mean that y contains more rational numbers than x.

Another related advantage of Dedekind cuts is that the least upper bound property is relatively simple to establish. Indeed, suppose B is a subset of $\mathbb{R}$ that is bounded from above, and let α be the union of all the elements (i.e., cuts) in B. One can then show in a fairly straightforward manner that this union α is itself a cut, and moreover, it is the least upper bound of B.

Exercise 5. *Show that $\mathbb{R}$ satisfies the least upper bound property.*

The main disadvantage of the Dedekind cut approach is that establishing the algebraic operations for the real numbers is a rather cumbersome process. There are other ways of constructing the real numbers such that this task is easier, but there, the least upper bound property is less apparent.

2.2. **Completeness.** Another way to think of the real numbers "filling in the holes" is via the notion of "completeness". Consider the infinite expansion

$$\sqrt{2} = 1.41421\ldots.$$

One can approximate $\sqrt{2}$ as closely as one would like using only rational numbers. For example, using decimal expansions, one can do this with the sequence

$$1, \quad 1.4, \quad 1.41, \quad 1.414, \quad \ldots.$$

A similar argument works for any real number. As a result of this, we say that the rational numbers form a *dense* subset of the real numbers.

Thus, one can see the real numbers as "filling in the missing slots not covered by the rationals". However, another question one can pose is whether the real number "fills in all the available missing slots". This is where the notion of completeness comes into play. Here, we shall show that the real number line is "complete", in the sense that all potential "slots" are in fact filled in.

To be much more precise, we say that a sequence $x_1, x_2, \ldots$ of real numbers is called a *Cauchy sequence* iff the following condition holds:

> For any positive real number ϵ, there exists a natural number N_0 such that $|x_n - x_m| < \epsilon$ for all natural numbers n, m satisfying $n, m \geq N_0$.

In other words, the x_n's become arbitrarily close to each other as n becomes arbitrarily large. Intuitively, since the x_n's bunch infinitely closer and closer together, we can think of Cauchy sequences as sequences that "approximate something".

In general, we say that a space is *complete* iff all Cauchy sequences in that space actually converge within the space. [7] Our goal here is then to show that the real number line $\mathbb{R}$ is complete. In other words, every Cauchy sequence in $\mathbb{R}$ – i.e., every sequence meant to "approximate something" – does in fact approximate an actual real number. From a more visual perspective, one can think of this as the real numbers "having no potential holes".

Remark. *This notion of completeness is also remarkably useful in other spaces besides the real numbers. For example, when solving differential equations, one often uses some iteration process to generate a sequence of "approximate solutions". In order to obtain an actual solution from these approximate solutions, one can sometimes rely on completeness. By showing that these approximate solutions form a Cauchy sequence in some complete space, then this sequence must converge to something, which one then shows is the actual solution.*

One usually shows the completeness of the real line using a general argument that can be adapted to many other types of spaces. [8] Here, however, we accomplish this goal specifically for the real numbers in a more elementary fashion, using the least upper bound property (again!). The main technical mechanism, besides the least upper bound property itself, is the following result.

Exercise 6. *Let (x_n) be a bounded nondecreasing sequence of real numbers, i.e.,*

$$x_1 \leq x_2 \leq x_3 \leq \cdots \leq M, \qquad M \in \mathbb{R}.$$

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[7] As given, this is of course too ambiguous of a statement. To be rigorous, one needs to define a formal notion of convergence and limits on this "space".

[8] See, e.g., [1].

Show that this sequence in fact converges. (Hint: Consider the least upper bound of the x_n's, and show that this is in fact the limit.) Similarly, show that a bounded nonincreasing sequence of real numbers also converges.

Suppose now that $x_1, x_2, \dots$ is a Cauchy sequence of real numbers. Since the x_n's must all eventually "cluster together" near some point, then the set of points $\{x_1, x_2, \dots\}$ is in fact bounded (can you show this?). As a result, by the least upper bound property, we can define the following numbers:

$$M_n = \sup\{x_n, x_{n+1}, x_{n+2}, \dots\}, \qquad m_n = \inf\{x_n, x_{n+1}, x_{n+2}, \dots\}, \qquad n \in \mathbb{N}.$$

Note that $m_n \leq M_n$ for each n. As the M_n's form a nonincreasing sequence, then by the preceding exercise, the M_n's have a limit

$$M = \limsup_n x_n = \lim_n M_n.$$

By a completely analogous reasoning, the m_n's also have a limit

$$m = \liminf_n x_n = \lim_n m_n.$$

We wish to use the Cauchy sequence property to show that $M = m$. Given $\epsilon > 0$, then by definition, there is some N_0 such that $|x_n - x_m| < \epsilon$ for all $n, m \geq N_0$. As a result, it follows that $|M_n - m_n| \leq \epsilon$ for any $n \geq N_0$. Our desired equality $M = m$ follows immediately from the above.

We want to show that *$M = m$ is in fact the limit of the x_n's.* For this, we can simply apply a "squeeze theorem" argument. By definition, we have

$$m_n \leq x_n \leq M_n, \qquad n \in \mathbb{N}.$$

Since the m_n's and M_n's both converge to the same element $M = m$, then the x_n's must also converge to this element. This proves that $\mathbb{R}$ is indeed complete.

To conclude, we remark that we can also use completeness in order to give an alternate construction of the real numbers. For this, we define the real number line $\mathbb{R}$ to be the set of all Cauchy sequences of rational numbers, with the additional caveat that two such sequences (x_n) and (y_n) are "the same" iff

$$\lim_{n \nearrow \infty} |x_n - y_n| = 0,$$

that is, they have the same limit. Intuitively, we identify a "real number x" with the class of sequences of rational numbers that converge to x.

One can show (after some considerable effort) that this construction is essentially equivalent to the Dedekind cut method. Algebraic operations are easier to describe in this setting, since one can easily add and multiply sequences element-wise. On the other hand, the least upper bound property is more difficult to establish.

Finally, we note that this Cauchy sequence "completion" approach can be directly generalized to other spaces besides the real numbers. Indeed, this method provides a rather generalized template for "filling the holes" in spaces.

References

1. W. Rudin, *Principles of mathematical analysis*, McGraw-Hill, Inc., 1976.
2. *Wikipedia: Proof that π is irrational*, `http://en.wikipedia.org/wiki/Proof_that_pi_is_irrational`, 2012.