SOLVING DIFFERENTIAL EQUATIONS

ARICK SHAO

1. Solving Ordinary Differential Equations

Consider first basic algebraic equations with an unknown, e.g.,

$$3(x + 2) = 18, \qquad 3x^2 = \cos x.$$

Such equations model some scenario in which one wishes to solve for some numerical value, represented by x, given some properties satisfied by x.

Let us now complicate our situation. The above equations contained only algebraic operations: addition, multiplication, exponentiation, and trigonometric operations. In many cases, one would also like to add *differential* or *integral* operations to the mix. For example, consider now the equation

$$y' = y,$$

which says that *the derivative of y is equal to y itself.* In contrast to the previous algebraic equations, where we were solving for a numerical value x, here we are solving for a *function* y of a single variable, say t. [1]

With some intuition from basic calculus, we can essentially guess the solutions of the above differential equation. For example, the functions

$$y_1(t) = e^t, \qquad y_2(t) = 10e^t$$

satisfy the equation $y' = y$. In fact, any function of the form

$$y(t) = Ce^t, \qquad C \in \mathbb{R},$$

is a solution to this differential equation. [2]

In many cases, we do not merely want a solution. We want to also establish that such a solution is unique, i.e., that it is the *only* solution to the equation. In order to tweak our basic example above so that we have unique solutions, the natural step is to convert it to an *initial value problem.* For example, if we wish to solve

$$y' = y, \qquad y(0) = 1,$$

that is, we wish to find a function y whose derivative is itself and its value at 0 is 1, then this problem actually has the unique solution $y(t) = e^t$.

Differential equations are used for modeling many real world phenomena, such as the motion of particles and population dynamics. One common and basic example

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[1] When y is a function of a single variable, we refer to such equations as *ordinary differential equations.* If we are dealing with functions of several variables, then such equations are called *partial differential equations.* The study of such partial differential equations comprises a major area of research in both pure and applied mathematics.

[2] In fact, these are *all* the solutions to this equation.

is the motion of a block attached to a spring. If one ignores frictional forces, then the motion of the block is described by *Hooke's Law*:

$$my'' = -ky.$$

Here, the unknown function $y = y(t)$ represents the position of the block at time t, while the positive constants m and k represent the mass of the block and the strength of the spring. We set $y = 0$ to represent the "equilibrium position", while the regions $y < 0$ and $y > 0$ represent positions in which the spring is compressed and stretched, respectively. Recalling the basic principles of Newtonian mechanics, then the above equation states that the force exerted on the block by the spring (i.e., the mass times the acceleration of the block) is toward the central "equilibrium position" $y = 0$. Moreover, the strength of this force is directly proportional to the distance the block is away from the equilibrium position.

From inspection, one sees that the solutions of this equation are of the form

$$y(t) = C_1 \cos \left(\sqrt{\frac{k}{m}} t \right) + C_2 \sin \left(\sqrt{\frac{k}{m}} t \right).$$

If we also require initial conditions for $y(0)$ and $y'(0)$, then one has a unique solution. In other words, absent frictional forces, the block oscillates indefinitely about $y = 0$, with a frequency determined by k and m. This is called *simple harmonic motion.*

Now, for the example equations mentioned above, one can find explicit simple solutions with relative ease. What if one presents a nastier equation, for instance,

$$y' = y^2 + \sin y, \qquad y(0) = 1?$$

We can no longer find explicit solutions constructed using only elementary functions, as before. However, even without the simple answers, could we still establish that solutions exist? Could we also show that such solutions are unique? The answer to these questions can often be reached using the "analytic toolbox" that we have developed by extending our understanding of limits and continuity.

1.1. **Existence and Uniqueness.**

We now show how one can prove the existence and uniqueness of solutions in general. Consider the initial value problem

$$y' = f(y), \qquad y(0) = c.$$

Here, y is the unknown function that we wish to find. As our method for solving such differential equations will be quite general, it will not be any more difficult to solve this general equation as opposed to something more specific.

In this text, we wish to prove the following result.

Theorem 1.1. *Suppose f is differentiable, and suppose f' is a bounded function. Then, there exists some time $0 < T < \infty$, depending on f, such that the above initial value problem has a unique solution $y : [0, T] \rightarrow \mathbb{R}$.*

Remark. *We actually made many more assumptions in Theorem 1.1 than necessary, for simplicity. However, one can make the result even more general.*

(1) *One does not require f' to be bounded everywhere. Such global bounds are overkill, as mere local bounds for f' will suffice. In particular, the existence and uniqueness result is still valid whenever f is any polynomial function.*
(2) *In fact, we need not assume f is differentiable at all. The conclusions of Theorem 1.1 still hold if this differentiability hypothesis is replaced by a weaker "Lipschitz continuity" condition.*

While the above points are extremely important for theoretical purposes, here we will focus only on the simplified version presented in Theorem 1.1, in order to avoid a multitude of technical details in the proof.

The strategy for solving this equation is via an iteration scheme. We begin with an arbitrary "initial" function. In each step of the iteration process, we produce a better approximation to the actual solution than the one before. By continuing this indefinitely, we obtain a sequence of successively closer approximations.

The final step is to advance from our sequence of approximations to an actual solution of the equation. To accomplish this, we will need our general notions of completeness and limits. In particular, we show that this sequence of approximations is a Cauchy sequence in the metric space $C[0,T]$, with the distance function

$$d(f,g) = \sup_{x\in[0,T]} |f(x) - g(x)|.$$

Since $C[0,T]$ is complete, then this sequence of approximations has a limit, which we can then show is the actual solution to the differntial equation.

1.2. **The Iteration Process.** We now describe in detail the iteration scheme. The first step is to restate our differential equation as the equivalent integral equation,

$$y(t) = c + \int_0^t f(y(s))ds.$$

The integral equation follows from the differential equation, and vice versa, by the fundamental theorem of calculus.

Consider now the following function:

$$\Phi : C[0,T] \to C[0,T], \qquad \Phi(y) = c + \int_0^t f(y(s))ds.$$

The precise value of T will be determined later. Then, a function y solves the desired integral (or differential) equation if and only if y is a *fixed point* of Φ, i.e.,

$$\Phi(y) = y.$$

Thus, we can restate our integral equation as a fixed point problem.

We construct our iteration as follows:

- The "initial function" y_1 can be any continuous function. For example, we can begin with the constant function, $y_1 \equiv c$.
- Define the next iterate y_2 by applying the integral equation to y_1:

$$y_2 = \Phi(y_1), \qquad y_2(t) = c + \int_0^t f(y_1(s))ds.$$

- One continues this inductively. Given an iterate y_n, we define

$$y_{n+1} = \Phi(y_n), \qquad y_{n+1}(t) = c + \int_0^t f(y_n(s))ds.$$

Through this process, the y_n's become closer and closer to our final solution. In order for this to work, though, we need our final time T to be "sufficiently small".

The main step is the following estimate:

$$\begin{aligned} d(y_{n+1}, y_n) &= d(\Phi(y_n), \Phi(y_{n-1})) \\ &= \sup_{t\in[0,T]} \left| c + \int_0^t f(y_n(s))ds - c - \int_0^t f(y_{n-1}(s))ds \right| \end{aligned}$$

$$= \sup_{t\in[0,T]} \left| \int_0^t [f(y_n(s)) - f(y_{n-1}(s))] ds \right| .$$

At this point, we can apply a crude bound to the right-hand side. Indeed, for each such s, the integrand here is at most

$$\sup_{z\in[0,T]} |f(y_n(z)) - f(y_{n-1}(z))|.$$

Also, our domain $[0, t]$ has length at most T. As a result, we have the estimate

$$d(y_{n+1}, y_n) \leq T \sup_{z\in[0,T]} |f(y_n(z)) - f(y_{n-1}(z))|.$$

To control the right-hand side here, we must invoke the assumption that f' is bounded. To be more specific, we assume that $|f'(x)| \leq R$ for all x. For a fixed $z \in [0, T]$, we know from the mean value theorem that

$$|f(y_n(z)) - f(y_{n-1}(z))| \leq |f'(x^*)||y_n(z) - y_{n-1}(z)|$$

for some point x^* between $y_n(z)$ and $y_{n-1}(z)$. Therefore, by our boundedness assumption on f' and our definition of the $C[0, T]$-distance, we have

$$d(y_{n+1}, y_n) \leq TR \cdot d(y_n, y_{n-1}).$$

Now, if our time of existence T is small enough, i.e., if $T < 1/(2R)$, then

$$d(y_{n+1}, y_n) \leq \frac{1}{2} d(y_n, y_{n-1}).$$

In other words, the "distance" between any two adjacent iterates is at most half of that between the previous two adjacent iterates. Thus, very far into our iteration process, i.e., when n is large, then $d(y_{n+1}, y_n)$ must be extremely small.

1.3. **Completeness and Convergence.** From our iteration scheme, we have constructed a sequence of functions in the metric space $C[0, T]$ which clump closer and closer together. These functions form approximate solutions which edge successively closer to the actual solution of the differential equation. The next step is to show that this sequence does in fact converge in the way we intended. As mentioned before, the main mechanism we need is the notion of completeness.

Lemma 1.2. *The sequence $y_1, y_2, y_3, \dots$ is a Cauchy sequence in $C[0, T]$.*

Proof. Fix natural numbers m, n; without loss of generality, we can assume that $m < n$. By the triangle inequality satisfied by the metric d, then

$$d(y_m, y_n) \leq \sum_{k=m}^{n-1} d(y_{k+1}, y_k).$$

Now, for each k, we can by induction show that

$$d(y_{k+1}, y_k) \leq \frac{1}{2} d(y_k, y_{k-1}) \leq \frac{1}{4} d(y_{k-1}, y_{k-2}) \leq \dots \leq 2^{-k} d(y_1, y_0),$$

where we used the main property we have shown for our iteration scheme. Combining the above, then we see that

$$d(y_m, y_n) \leq d(y_1, y_0) \sum_{k=m}^{n-1} 2^{-k} < 2^{-m} d(y_1, y_0) \sum_{k=0}^{\infty} 2^{-k} \leq 2^{-m+1} d(y_1, y_0).$$

As a result, by choosing m and n to be sufficiently large, we can make $d(y_m, y_n)$ as small as we like. Consequently, the sequence (y_k) is Cauchy. □

Since our metric space $C[0,T]$ is complete, then the y_k's have a limit y (with respect to the metric d on $C[0,T]$). The last step is to show that y is a solution to the differential equation, that is, y is a fixed point of Φ.

Lemma 1.3. *The following continuity condition holds:*

$$\Phi(y) = \lim_{n\to\infty} \Phi(y_n).$$

Exercise 1. *Prove Lemma 1.3!*

With Lemma 1.3 in hand, we now have that

$$\Phi(y) = \lim_{n\to\infty} \Phi(y_n) = \lim_{n\to\infty} y_{n+1} = y.$$

Moreover, since $y_k(0) = c$ for every k, then $y(0) = c$ as well, so that y is indeed the solution to our initial value problem!

We have now established that a solution y to our initial value problem *exists*. We have not yet shown that this solution is unique. To complete this final part, suppose w is another solution to our initial value problem. Then, by a process that is analogous to our estimate for our iteration scheme, we can show

$$d(y,w) = d(\Phi(y),\Phi(w)) \leq \sup_{t\in[0,T]} \int_0^t |f(y(s)) - f(w(s))| ds \leq \frac{1}{2} d(y,w).$$

The only way the above can hold is if $d(y,w) = 0$. As a result, $y = w$, and we have proved that our solution is in fact unique!

Exercise 2. *Prove the above bound, needed for the proof of uniqueness.*

This completes the proof of Theorem 1.1.

1.4. **Global Solutions?** Theorem 1.1 guarantees the existence and uniqueness of a *local solution* $y : [0,T] \to \mathbb{R}$ to our initial value problem, if T is sufficiently small. Can we push the solution further indefinitely? Can we find a *global solution* to our initial value problem? In other words, is there a (unique) solution of the form

$$y : [0,\infty) \to \mathbb{R}, \qquad y' = f(y), \qquad y(0) = c?$$

In the setting of Theorem 1.1, the answer is "yes", due to f' being globally bounded. In our iteration scheme, we needed that the "time of existence" T be sufficiently small with respect to our bound R for f'. Now, let us apply Theorem 1.1 again, but with initial data $c_1 = y(T/2)$, rather than c. This once again yields a local solution on a time interval of length T.

Since these local solutions must be unique, then the first half of this new solution (with initial data c_1) must coincide with our original solution y (with initial data c) on the interval $[T/2, T]$. Thus, we can "glue" these two solutions together and obtain a new longer local solution $y_1 : [0, 3T/2] \to \mathbb{R}$.

Repeating this process, now with initial data $c_2 = y(T)$ yields an even longer solution $y_2 : [0, 2T] \to \mathbb{R}$. Iterating indefinitely and "gluing" appropriately, we obtain a (unique) global solution $y_\infty : [0,\infty) \to \mathbb{R}$ to our initial value problem!

References

1. W. Rudin, *Principles of mathematical analysis*, McGraw-Hill, Inc., 1976.
2. J. Stewart, *Multivariable calculus*, 7 ed., Brooks/Cole, 2012.