SET THEORETIC PARADOXES

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1. Set Theoretic Paradoxes

When one first discusses formal mathematics and rigorous proofs, the notion of sets is usually intentionally left vague, or even undefined. Although sets are usually considered to form the foundations of all mathematics, one's attention is generally focused on higher-level mathematical concepts, such as the properties of number systems, algebraic structures, topological structure, and so on.

Thus, one is usually content to "define" a "set" ambiguously as a "collection of things". In practice, these "sets" can be described in various ways:

(1) The elements of a set can be directly listed, e.g.,

$$\{1, 2, 3\}, \qquad \{2, 5, 7\}, \qquad \{3, 6, 9, \dots, 303\}.$$

(2) One can also enumerate infinite sets, according to some understood pattern:

$$\mathbb{N} = \{1, 2, 3, \dots\}, \qquad \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

(3) More generally, sets can be described using some *membership criterion.* For example, we can write

$$\{n \mid n \text{ is a prime number}\}$$

to mean the set of all n for which n is a prime number, or more succintly, the set of all prime numbers. Another example is the set

$$\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$$

of all real numbers between 0 and 1, inclusive.

We focus on method (3) of our ways of constructing sets. In one's first encounter with set theory, one typically allows any logical true/false statement as a membership criterion. More specifically, one can generally define a set by

$$\{x \mid \varphi(x)\},$$

where $\varphi(x)$ is such a membership criterion.

In our first example, the membership criterion is "n is a prime number". Any arbitrary "element" n is a member of this set if and only if n satisfies this membership criterion, i.e., when n is a prime number. As a result, the set

$$\{n \mid n \text{ is a prime number}\}$$

consists of precisely all the prime numbers. The second example is analogous, and the membership criterion $0 \leq x \leq 1$ produces the closed interval $[0, 1]$. [1]

The above informal approach, in which sets are defined rather loosely, is generally referred to as *naive set theory.* The development of this "theory" of sets is

[1] To be completely accurate, the condition $x \in \mathbb{R}$ is also a part of the membership criterion, so that the full membership criterion $\varphi(x)$ is in fact "x is a real number, and $0 \leq x \leq 1$.

typically attributed to Georg Cantor, in the late 1800's. However, mathematicians soon discovered that such lax foundations produced crippling logical paradoxes that undermined the entire theory. As all of mathematics and mathematical proofs are based on logical reasoning and arguments, such foundational paradoxes presented crises that threatened to invalidate all of mathematics! Here, we discuss some of the most basic set theoretic paradoxes, and we briefly discuss how these were resolved by developing more formal, or axiomatic, mathematical foundations.

1.1. **Berry's Paradox.** The first paradox we discuss here, which is attributed to G. G. Berry, is purely linguistic. Consider the set

$$B = \{n \in \mathbb{N} \mid n \text{ can be defined in fifty or less (English) words}\}.$$

Since there are only finitely many English words in the universe, there are only finitely many ways to arrange up to fifty of these finitely many words. Consequently, our set B must be finite.

Being a finite subset of the natural numbers, then B as a largest element. In other words, there is a largest natural number N that can be defined in fifty or less words. Consider now the number $N + 1$, which, due to our description of N, cannot be defined using at most fifty words. But, we can describe $N + 1$ in English as "one greater than the largest natural number that can be defined in fifty or less words!" Therefore, we have established a contradiction.

The heart of this paradox lies in the ambiguities of natural languages. The criterion "n can be defined in fifty or less words" is an example of a statement in English that is not formally and rigorously well-defined. The introduction of such ambiguous statements into our naive set theory introduces logical problems which develop into contradictions, as shown above.

In mathematics, Berry's paradox is resolved by taking much more care in crafting our formal language. Indeed, one restricts oneself only to logically well-defined statements. For example, a set membership statement (e.g., $x \in A$) is such a basic unambiguous statement. More complex well-defined statements can be constructed from more basic ones using logical connectives ("and", "or", "if . . . , then . . . ", etc.) or quantifiers ("for all . . . ", "there exists . . . "). In generally, by being far more cautious with the mathematical language, in particular distancing oneself from ambiguous natural language statements, one can prevent linguistic contradictions of the above type.

1.2. **Russell's Paradox.** The following paradox, named after Bertrand Russell, is more subtle, as it does not require a faulty adaptation of logical language to be observed. In contrast, this paradox can be constructed using purely logical statements. The defect here is in the set theory itself.

Using our naive set theory, we can construct the set

$$R = \{x | x \notin x\},$$

i.e., the set of all things x in our mathematical universe which are not elements of itself. At first, this definition seems innocuous enough, but trouble quickly arises when we consider R itself.

One can now ask: is R itself an element of R? The answer, of course, is either "yes" or "no". We can examine each case separately.

- If R is an element of R, that is, if $R \in R$, then R must satisfy the above membership criterion for R – thus, $R \notin R$. This is of course a contradiction, so we can conclude from here that R cannot be an element of R.
- Suppose now that R is not an element of R, that is, $R \notin R$. Then, R satisfies the membership criterion for R, and hence $R \in R$. This is yet another contradiction, and this implies that R must be an element of R.

The two branches of arguments show that R both *cannot be an element of* R and *cannot not be an element of* R. We now have obtained a disastrous and epic contradiction, from which we cannot recover.

What creates this contradiction is the contruction of the set R, which destroys everything. To resolve the paradox, then, one must restrict what one can define as a "set". In particular, one must replace the loose definition of a set from naive set theory by a more restrictive notion, so that one cannot adopt membership criteria that result in disastrous sets, such as R.

In response to Russell's paradox, mathematicians developed *axiomatic set theories*, which, through various "reasonable" axioms, formally and precisely specify what can and cannot be sets. [2] A large part of this endeavor consists of determining which membership criterion actually define sets, and which are invalid. Without engaging in details, the general idea is to avoid constructing sets which are "too large". The object R described above is such an example of something that is "too large to be a set". In particular, one important consequence of Russell's paradox is that *there cannot be a set of all sets* (as this is definitely "too large").

1.3. **"Paradoxes" in Axiomatic Set Theories.** By constructing axiomatic set theories, we have essentially prevented both Berry's and Russell's paradoxes from occurring. However, such "paradoxes" can still form informative statements in these axiomatic theories, even though they have been resolved and hence no longer create contradictions. Indeed, in the contexts of these axiomatized settings, these former paradoxes now become proofs of negative statements.

For example, consider again Russell's paradox, but now within an axiomatized set theory, which determines what can and cannot be sets. Then, the above reasoning regarding R shows that $\{x \mid x \in x\}$ does not define a set. More accurately, *there is no set which contains all objects x which are not elements of itself.* The previous demonstration of Russell's paradox is now a proof by contradiction of the above negative statement!

One can also construct analogues of Berry's paradox using completely formal and logical statements. Again, these are no longer contradictions, but they can be used to show certain situations are impossible. For example, such a technique has been used to prove Gödel's incompleteness theorem!

References

1. H. B. Enderton, *Elements of set theory*, Academic Press, 1977.
2. *Wikipedia: Berry paradox*, `http://en.wikipedia.org/wiki/Berry_paradox`, 2012.
3. *Wikipedia: Russell's paradox*, `http://en.wikipedia.org/wiki/Russell's_paradox`, 2012.

[2] The axiomatic set theories most widely used as the foundations of mathematics are the *Zermelo-Fraenkel theory with the axiom of choice* (ZFC) and the *von Neumann-Bernays-Gödel theory* (NBG).