Fall 2012 MAT334 Exam 1
Solutions (LEC0101)

Problem 1. *Let C denote the bottom half of the positively oriented unit circle with center i, i.e., counterclockwise from $i-1$ to $i+1$. Evaluate*

$$\int_C (z-i)^{-3} dz.$$

Solution. The curve C can be parametrized as

$$\gamma(t) = i + e^{it}, \qquad \gamma'(t) = ie^{it}, \qquad -\pi \leq t \leq 0.$$

Thus, the line integral becomes

$$\int_C (z-i)^{-3} dz = \int_{-\pi}^{0} (\gamma(t)-i)^{-3}\gamma'(t) dt = \int_{-\pi}^{0} (i+e^{it}-i)^{-3} ie^{it} dt = i\int_{-\pi}^{0} e^{-2it} dt.$$

The complex-valued integral on the right-hand side can be computed directly:

$$\int_C (z-i)^{-3} dz = \frac{i}{-2i}\, e^{-2it}\big|_{-\pi}^{0} = -\frac{1}{2}(e^0 - e^{2\pi i}) = 0.$$

Remark. One could also take t to be the interval $\pi \leq t \leq 2\pi$. Also, one could expand e^{it} as $\cos t + i \sin t$. The final answer, of course, remains unchanged.

Problem 2. *Let D_1 and D_2 be open subsets of $\mathbb{C}$. Prove that their intersection, $D_1 \cap D_2$, is also open.*

Solution. Given $z \in \mathbb{C}$ and $r > 0$, let $B_z(r)$ denote the open disk with center z and radius r.

Suppose $z \in D_1 \cap D_2$. Since $z \in D_1$ and D_1 is open, there is some $r_1 > 0$ such that $B_z(r_1) \subseteq D_1$. Similarly, since $z \in D_2$, there is some $r_2 > 0$ with $B_z(r_2) \subseteq D_2$. Letting $r = \min(r_1, r_2)$, then $B_z(r)$ is contained in both D_1 and D_2, that is, $B_z(r) \subseteq D_1 \cap D_2$.

This shows that every point of $D_1 \cap D_2$ is an interior point. Thus, by definition, $D_1 \cap D_2$ is open.

Remark. There are certainly other correct proofs, but the above method is the shortest and most direct.

Problem 3. *Find all solutions z of the equation*

$$z^5 = \sqrt{3} - i.$$

Solution. In polar coordinates,

$$z^5 = \sqrt{3} - i = 2e^{(-\frac{\pi}{6}+2\pi k)i}$$

for any integer k. Taking the fifth root of the above yields

$$z = \sqrt[5]{2} \cdot e^{\frac{1}{5}(-\frac{\pi}{6}+2\pi k)i}.$$

Eliminating all nonunique (repeating) values, we obtain five solutions,

$$z = \sqrt[5]{2} \cdot e^{(-\frac{\pi}{30} + \frac{2\pi k}{5})i}, \qquad k = 0, 1, 2, 3, 4.$$

Remark. One could also write

$$\sqrt{3} - i = 2e^{(\frac{11\pi}{6} + 2\pi k)i},$$

and with this, the (same) five solutions are

$$z = \sqrt[5]{2} \cdot e^{(\frac{11\pi}{30} + \frac{2\pi k}{5})i}, \qquad k = 0, 1, 2, 3, 4.$$

Problem 4. *Consider the following limit:*

$$\lim_{z \to 2} \frac{\bar{z} - 2}{z - 2}.$$

If the limit exists, prove it (using δ-ϵ or some other rigorous means). If the limit does not exist, show why not.

Solution. **The limit does not exist!**

On the horizontal line through 2 ($\operatorname{Im} z = 0$, i.e., $z = x \in \mathbb{R}$),

$$\frac{\bar{z} - 2}{z - 2} = \frac{x - 2}{x - 2} = 1,$$

while on the vertical line through 2 ($\operatorname{Re} z = 2$, i.e., $z = 2 + iy$),

$$\frac{\bar{z} - 2}{z - 2} = \frac{-iy}{iy} = -1.$$

Thus, as $z \to 2$, the function $(\bar{z} - 2)/(z - 2)$ approaches two different values, 1 from the horizontal line $\operatorname{Im} z = 0$, and -1 from the vertical line $\operatorname{Re} z = 2$. Thus, the limit in the problem statement cannot possibly exist.

Remark. This is essentially the example discussed in class. If we let $w = z - 2$, i.e., we shift the picture left by 2, then the limit becomes

$$\lim_{w \to 0} \frac{\bar{w}}{w},$$

which was shown to not exist (the argument was the same as in the solution).

Problem 5. *For which $z \in \mathbb{C}$ does the series*

$$\sum_{n=0}^{\infty} |z^n + z^{n+1}|$$

converge?

Solution. The partial sums of the series can be written as

$$\sum_{n=0}^{m} |z^n + z^{n+1}| = \sum_{n=0}^{m} |1+z||z^n| = |1+z| \sum_{n=0}^{m} |z|^n.$$

The right-hand side is simply a constant, $|1+z|$, times (the partial sum of) the geometric series for $|z|$.

Suppose first that $z \neq -1$, so that the constant $|1+z|$ is nonzero. The, the series $\sum_{n=0}^{\infty} |z^n + z^{n+1}|$ converges if and only if the geometric series $\sum_{n=0}^{\infty} |z|^n$ converges. Since the geometric series converges if $|z| < 1$ and diverges if $|z| \geq 1$, then the same holds for $\sum_{n=0}^{\infty} |z^n + z^{n+1}|$.

On the other hand, if $z = -1$, then each term of the series satisfies

$$|z^n + z^{n+1}| = |1+z||z|^n = 0,$$

and hence

$$\sum_{n=0}^{\infty} |z^n + z^{n+1}| = \sum_{n=0}^{\infty} 0 = 0.$$

Thus, **the series converges if $|z| < 1$ or $z = -1$ and diverges otherwise.**

Remark. One can also apply the ratio test:

$$\frac{|z^{n+1} + z^{n+2}|}{|z^n + z^{n+1}|} = \frac{|1+z||z|^{n+1}}{|1+z||z|^n} = |z|.$$

This implies that the series converges when $|z| < 1$ and diverges when $|z| > 1$. The remaining case $|z| = 1$ still has to be tested directly, though.