Fall 2012 MAT334 Exam 2 Solutions (LEC0101)

Problem 1. *Let the curve γ be the positively-oriented boundary of the square with corners at $2+2i$, $-2+2i$, $-2-2i$, and $2-2i$. Evaluate the following:*

$$\int_\gamma \frac{e^z}{z^2+2z-3} dz.$$

Solution. First, the integrand can be factored as

$$\frac{e^z}{z^2+2z-3} = \frac{e^z}{(z+3)(z-1)}.$$

Thus, the poles of the integrand lie at $z_0 = -3$ and $z_0 = 1$. Since -3 lies outside γ and 1 lies within γ, we can write the above integral as

$$\int_\gamma \frac{e^z}{z^2+2z-3} dz = \int_\gamma \frac{\frac{e^z}{z+3}}{z-1} dz = \int_\gamma \frac{f(z)}{z-1} dz,$$

where this function f is holomorphic *in some convex domain D that contains γ and its interior.* As a result, we can apply the Cauchy formula:

$$\int_\gamma \frac{e^z}{z^2+2z-3} dz = 2\pi i \cdot f(1) = \frac{1}{2} e\pi i.$$

Problem 2. *For what $z \in \mathbb{C}$ is the function $f(z) = |z|^2$ (complex) differentiable? For what $z \in \mathbb{C}$ is f not differentiable?*

Solution. The most systematic method is with the Cauchy-Riemann equations. Letting

$$f(z) = u(x,y) + iv(x,y), \qquad z = x + iy,$$

we see that $u(x,y) = x^2 + y^2$ and $v(x,y) = 0$. Thus,

$$\partial_x u(x,y) = 2x, \qquad \partial_y u(x,y) = 2y, \qquad \partial_x v(x,y) = 0, \qquad \partial_y v(x,y) = 0.$$

Since u and v are both continuously differentiable, f is complex differentiable if and only if $\partial_x u = \partial_y v$ and $\partial_x v = -\partial_y u$. This is true if and only if $x = y = 0$, i.e., when $z = 0$. Thus, f is differentiable when $z = 0$, and f is not differentiable for any other z.

Remark. One can also solve this using the definition of differentiability. Fix $z \in \mathbb{C}$, and consider the limit

$$\begin{aligned} \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h \to 0} \frac{|z+h|^2 - |z|^2}{h} \\ &= \lim_{h \to 0} \frac{z\bar{h} + \bar{z}h}{h} \\ &= z + \bar{z} \lim_{h \to 0} \frac{\bar{h}}{h}. \end{aligned}$$

From an example in class (or from the first midterm!), the limit of $\bar{h}/h$ as $h \to 0$ does not exist. Thus, the derivative $f'(z)$ exists if and only if $\bar{z} = 0$, i.e., if and only if $z = 0$.

Problem 3. *Suppose f and g are holomorphic functions such that:*

- *f has a zero of order $m > 0$ at $z_0 \in \mathbb{C}$.*
- *g has a pole of order m at z_0.*

Prove that $f \cdot g$ can be analytically extended to a nonzero value at z_0.

Solution. Since f has a zero of order m at z_0, we can write

$$f(z) = (z - z_0)^m F(z),$$

where F is some holomorphic function (in particular at z_0) that is nonzero at z_0. Similarly, since g has a pole of order m at z_0, we can write

$$g(z) = (z - z_0)^{-m} G(z),$$

where G is some holomorphic function (near and at z_0) that is nonzero at z_0.

As a result, $f(z)g(z) = F(z)G(z)$ when $z \neq z_0$, and the right-hand side provides an analytic extension of $f \cdot g$ to z_0. Moreover, since neither F nor G vanish at z_0, then $F(z_0)G(z_0) \neq 0$.

Problem 4. *Let*

$$f(z) = \frac{(z+2)\sin z}{(e^z - 1)^2}.$$

Show that f has a pole of order 1 at $z = 0$, and then compute $\operatorname{Res}(f; 0)$.

Solution. Note the following:

- $z + 2$ is an entire function, whose value at $z = 0$ is 2.
- From the Taylor series for sin, we can write

  $$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!} = z g(z),$$

  where $g(z)$ is entire, and where $g(0) = 1$.
- From the Taylor series for exp, we can write

  $$e^z - 1 = \sum_{n=0}^{\infty} \frac{z^n}{n!} - 1 = \sum_{n=1}^{\infty} \frac{z^n}{n!} = z \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} = z h(z),$$

  where $h(z)$ is entire, and where $h(0) = 1$.

As a result, we can write

$$f(z) = \frac{z}{z^2} \frac{(z+2)g(z)}{[h(z)]^2} = \frac{1}{z} \frac{(z+2)g(z)}{[h(z)]^2}.$$

Since the fraction $(z+2)g(z)/[h(z)]^2$ is entire and nonzero at $z = 0$, it follows from this representation that f has a pole of order 1 at $z = 0$.

Finally, to compute the residue, since the pole at $z = 0$ has order 1,

$$\operatorname{Res}(f; 0) = zf(z)|_{z=0} = \frac{(0+2)g(0)}{[h(0)]^2} = 2.$$

Remark. There is no need to deal with the full Taylor series for $\sin z$ and $e^z - 1$. For example, one can also recall that

$$g(0) = \sin' 0 = 1, \qquad h(0) = \frac{d}{dz}(e^z - 1)|_{z=0} = 1.$$

The above formula for the residue was given in class. This can also be derived directly. Since $F(z) = (z+2)g(z)/[h(z)]^2$ is entire, it has a power series

$$F(z) = \sum_{n=0}^{\infty} c_n z^n.$$

As a result,

$$f(z) = z^{-1}F(z) = c_0 z^{-1} + \sum_{n=0}^{\infty} c_{n+1} z^n,$$

and hence

$$\operatorname{Res}(f; 0) = c_0 = F(0) = 2.$$

The residue can also be computed directly using the Cauchy formula.

Problem 5. *Let γ be any smooth piecewise continuous curve from $-i$ to i that does not pass through the* positive real axis $[0, \infty)$. *Compute*

$$\int_\gamma \frac{1}{z} dz.$$

(Hint: The answer is **not** $+\pi i$*).*

Solution. The main idea is to use the fundamental theorem of calculus, since z^{-1} has the logarithm as its antiderivative. The caveat, however, is that this log has to be holomorphic wherever the curve γ is defined. Since γ is assumed to not cross the positive real axis, we should choose a branch of log which fails to be holomorphic (and continuous) on the positive real axis. For example, we can use the branch

$$L(z) = \ln z + i \arg z, \qquad 0 \leq \arg z < 2\pi.$$

Using L and the fundamental theorem of calculus,

$$\begin{aligned}\int_\gamma \frac{1}{z} dz &= \int_\gamma L'(z) dz \\ &= L(i) - L(-i) \\ &= \left(\ln i + \frac{\pi}{2} i\right) - \left[\ln(-i) + \frac{3\pi}{2} i\right] \\ &= -\pi i.\end{aligned}$$

Remark. Since the principal logarithm (Log) is not continuous on the *negative* real axis, where γ crosses, we cannot use this as the "correct" antiderivative of z^{-1}. Indeed, using Log yields the incorrect answer πi.