MAT 334 Practice Exam 1 Solutions

Problem 1. *Let L denote the line segment from $3-2i$ to -1. Evaluate*

$$\int_L z dz.$$

Solution. The first step is to parametrize the curve. The simplest such parametrization is the one from precalculus/calculus:

$$\gamma(t) = -1 \cdot t + (3-2i) \cdot (1-t), \qquad 0 \le t \le 1.$$

Since $\gamma'(t) = -1 - (3-2i) = -4+2i$, then

$$\begin{aligned}
\int_L z dz &= \int_0^1 \gamma(t)\gamma'(t)dt \\
&= \int_0^1 [-t + (3-2i)(1-t)]dt \cdot (-4+2i) \\
&= (-4+2i) \int_0^1 [(-4+2i)t + (3-2i)]dt \\
&= (-4+2i)\left[\frac{1}{2}(-4+2i) + (3-2i)\right] \\
&= (-4+2i)(1-i) \\
&= -2+6i.
\end{aligned}$$

Problem 2. *Prove or disprove: if both $U \subseteq \mathbb{C}$ and $V \subseteq U$ are open, then*

$$U \setminus V = \{z \mid z \in U \text{ and } z \notin V\},$$

i.e., the set of all points in U which are not in V, is also open.

Solution. **The statement is false.** For example, if

$$U = \{z \mid |z| < 2\}, \qquad V = \{x \mid |z| < 1\},$$

then both U and V are open, and $V \subseteq U$. One can then see that

$$U \setminus V = \{z \mid 1 \le |z| < 2\},$$

i.e., the annulus with inner radius 1 and outer radius 2.

Now, any point z with $|z| = 1$ is an element of $U \setminus V$. However, this z is also a boundary point of $U \setminus V$ (intuitively, if you take any small step from z toward the origin, you are no longer on $U \setminus V$). Since $U \setminus V$ contains boundary points, then $U \setminus V$ cannot be open.

Problem 3. *Find all values of* 2^{-i}.

Solution. This can be computed directly:

$$2^{-i} = e^{-i(\log 2)} = e^{-i(\ln 2 + i \arg 2)} = e^{-i\ln 2 + 2\pi n} = e^{-i\ln 2} e^{2\pi n}.$$

Problem 4. *Does the limit*

$$\lim_{z\to 0} e^{\frac{1}{z}}$$

exist? Justify your answer. (Note: ∞ *is allowed to be a possible limit value.)*

Solution. **The limit does not exist.**

To see this, we need only check what happens to $e^{1/z}$ when $z \to 0$ from different directions. First, consider when $z \to 0$ along the positive x-axis, i.e., $z = x \in \mathbb{R}$ and $x \searrow 0$. Since $1/x \nearrow +\infty$ as $x \searrow 0$, we have

$$\lim_{x\to 0} e^{\frac{1}{x}} = \infty,$$

On the other hand, if $z \to 0$ instead along the negative x-axis, i.e., $z = x \in \mathbb{R}$ and $x \nearrow 0$, then $1/x \searrow -\infty$, and hence

$$\lim_{x\to 0} e^{\frac{1}{x}} = 0.$$

Remark. Also, notice that if $z \to 0$ along the imaginary axis, i.e., $z = iy \in i\mathbb{R}$ and $y \to 0$, then $e^{1/z} = e^{-i/y}$ oscillates infinitely many times along the unit circle, and the limit in this direction does not exist.

Problem 5. (Fall 2012, Midterm 1) *Consider, for any fixed* $z \in \mathbb{C}$, *the series*

$$\sum_{n=0}^{\infty} |z^n + z^{n+1}|.$$

For which z *does this series converge? Diverge?*

Solution. The partial sums of the series can be written as

$$\sum_{n=0}^{m} |z^n + z^{n+1}| = \sum_{n=0}^{m} |1+z||z^n|.$$

Suppose first that $z \neq -1$, so that the constant $|1+z|$ is nonzero. Then,

$$\sum_{n=0}^{m} |z^n + z^{n+1}| = |1+z| \sum_{n=0}^{m} |z^n| = |1+z| \sum_{n=0}^{m} |z|^n.$$

The right-hand side is simply a constant, $|1+z|$, times (the partial sum of) the geometric series for $|z|$. Thus, $\sum_{n=0}^{\infty} |z^n + z^{n+1}|$ converges if and only if the geometric series $\sum_{n=0}^{\infty} |z|^n$ converges. Since the geometric series converges if $|z| < 1$ and diverges if $|z| \geq 1$, then the same holds for $\sum_{n=0}^{\infty} |z^n + z^{n+1}|$.

On the other hand, if $z = -1$, then each term of the series satisfies

$$|z^n + z^{n+1}| = |1+z||z|^n = 0,$$

and hence

$$\sum_{n=0}^{\infty} |z^n + z^{n+1}| = \sum_{n=0}^{\infty} 0 = 0.$$

The series converges if $|z| < 1$ or $z = -1$ and diverges otherwise.

Remark. One can also apply the ratio test:

$$\frac{|z^{n+1} + z^{n+2}|}{|z^n + z^{n+1}|} = \frac{|1+z||z|^{n+1}}{|1+z||z|^n} = |z|.$$

This implies that the series converges when $|z| < 1$ and diverges when $|z| > 1$. The remaining case $|z| = 1$ still has to be tested directly, though.