MAT 334 Practice Exam 2 Solutions

Note: These are **detailed** *solutions to the practice exam questions. For the actual exam, you can (and should!) write considerably less, as long as you still show your work and capture the main ideas.*

Problem 1. (Fall 2012, Final Exam) *Does the following limit exist:*

$$\lim_{z\to 0} z^{-1}\operatorname{Log}(1-z)?$$

If so, find the limit; otherwise, show that the limit does not exist.

Solution. The most straightforward way is to recall the Taylor series:

$$\operatorname{Log}(1-z) = -\sum_{n=1}^{\infty}\frac{z^n}{n} = -\sum_{n=0}^{\infty}\frac{z^{n+1}}{n+1}, \qquad |z|<1.$$

Dividing by z yields

$$z^{-1}\operatorname{Log}(1-z) = -\sum_{n=0}^{\infty}\frac{z^n}{n+1}, \qquad |z|<1.$$

The right-hand side is well-defined and continuous at $z=0$; this is simply the first term of the infinite series. Therefore,

$$\lim_{z\to 0} z^{-1}\operatorname{Log}(1-z) = -\frac{1}{0+1} = -1.$$

Remark. Alternatively, since $\operatorname{Log}(1-z)$ and z are both analytic near $z=0$, and since both $\operatorname{Log}(1-z)$ and z are 0 at $z=0$, one can also apply l'Hopital's rule (which has the same proof for complex functions as for real functions):

$$\lim_{z\to 0}\frac{\operatorname{Log}(1-z)}{z} = \lim_{z\to 0}\frac{-\frac{1}{1-z}}{1} = -1.$$

Problem 2. *Compute the following line integral:*

$$\int_{|z-2|=1}\frac{dz}{z^2-2z+iz-2i}.$$

Assume that the curve in the line integral is positively oriented.

Solution. The goal is to apply Cauchy's formula, so we need to manipulate the integrand so that it is in the right form. For this, we factor,

$$\int_{|z-2|=1} \frac{dz}{z^2 - 2z + iz - 2i} = \int_{|z-2|=1} \frac{dz}{(z-2)(z+i)}.$$

Note the integrand is analytic everywhere, except at 2 and $-i$. Moreover, 2 is *inside* the circle $|z-2| = 1$, while $-i$ is *outside* the circle.

To explicitly connect the above to Cauchy's formula, we let D be some convex (or simply connected) domain which contains the circle $|z-2| = 1$ but does not contain $-i$. We take for our function

$$f : D \to \mathbb{C}, \qquad f(z) = \frac{1}{z+i},$$

which is analytic on D. Then, applying Cauchy's formula to f yields

$$\begin{aligned} \int_{|z-2|=1} \frac{dz}{z^2 - 2z + iz - 2i} &= \int_{|z-2|=1} \frac{f(z)}{z-2} dz \\ &= 2\pi i \cdot n(|z-2| = 1; 2) \cdot f(2) \\ &= 2\pi i \cdot f(2) \\ &= \frac{2\pi i}{2+i}. \end{aligned}$$

Problem 3. (Fall 2012, Midterm 2) *For which $z \in \mathbb{C}$ is the function $f(z) = |z|^2$ (complex) differentiable? For which $z \in \mathbb{C}$ is f not differentiable?*

Solution. The most systematic method is with the Cauchy-Riemann equations. Splitting f into its real and imaginary components,

$$f(z) = u(x, y) + iv(x, y), \qquad z = x + iy,$$

we see that $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$. Thus,

$$\partial_x u(x, y) = 2x, \qquad \partial_y u(x, y) = 2y, \qquad \partial_x v(x, y) = 0, \qquad \partial_y v(x, y) = 0.$$

Since u and v are both continuously differentiable, f is complex differentiable if and only if $\partial_x u = \partial_y v$ and $\partial_x v = -\partial_y u$. This is true if and only if $x = y = 0$, i.e., when $z = 0$. Thus, f is differentiable at $z = 0$, while f is not differentiable for any nonzero $z \in \mathbb{C}$.

Remark. One can also solve this using the definition of differentiability. Fix $z \in \mathbb{C}$, and consider the limit

$$\begin{aligned}\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} &= \lim_{h\to 0} \frac{|z+h|^2 - |z|^2}{h} \\ &= \lim_{h\to 0} \frac{z\bar{h} + \bar{z}h + h\bar{h}}{h} \\ &= \bar{z} + z \lim_{h\to 0} \frac{\bar{h}}{h}.\end{aligned}$$

From an example in the book, the limit of $\bar{h}/h$ as $h \to 0$ does not exist. Thus, the derivative $f'(z)$ exists if and only if $z = 0$.

Problem 4. (Fall 2012, Final Exam) *For the (analytic) function*

$$f(z) = \frac{1}{z^2 - 4z + 4},$$

find its power series representation about 0*, and find its radius of convergence.*

Solution. The key is to write the right-hand side in terms of something for which we know the power series expansion. More specifically, we write

$$f(z) = \frac{1}{(z-2)^2} = -\frac{d}{dz}\left(\frac{1}{z-2}\right).$$

Now, $1/(z-2)$ can be written as a geometric series:

$$\frac{1}{z-2} = -\frac{1}{2} \cdot \frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\sum_{n=0}^{\infty} \frac{z^n}{2^n}, \qquad \left|\frac{z}{2}\right| < 1.$$

Moreover, from the root test, e.g., one can see that the above series has radius of convergence $R = 2$ (this is also clear from the fact that the standard geometric series $(1-w)^{-1} = \sum_n w^n$ has radius of convergence 1).

Now, since any power series can be differentiated termwise, *without changing the radius of convergence*, then we obtain

$$f(z) = \frac{1}{2}\frac{d}{dz}\sum_{n=0}^{\infty} \frac{z^n}{2^n} = \sum_{n=1}^{\infty} \frac{nz^{n-1}}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{(n+1)z^n}{2^{n+2}},$$

with the radius of convergence once again being $R = 2$.

Problem 5. (Fall 2012, Midterm 2) *Let γ be any smooth piecewise continuous curve from $-i$ to i that does not pass through the* positive real axis $[0, \infty)$. *Compute*

$$\int_\gamma \frac{1}{z} dz.$$

(Hint: The answer is **not** $+\pi i$*).*

Solution. The main idea is to use the fundamental theorem of calculus, since z^{-1} has the logarithm as its antiderivative. The caveat, however, is that this log has to be holomorphic wherever the curve γ is defined. Since γ is assumed to not cross the positive real axis, we should choose a branch of log which fails to be holomorphic (and continuous) on the positive real axis. For example, we can use the branch

$$L(z) = \ln z + i \arg z, \qquad 0 \leq \arg z < 2\pi.$$

Using L and the fundamental theorem of calculus,

$$\begin{aligned}
\int_\gamma \frac{1}{z} dz &= \int_\gamma L'(z) dz \\
&= L(i) - L(-i) \\
&= \left(\ln i + \frac{\pi}{2} i\right) - \left[\ln(-i) + \frac{3\pi}{2} i\right] \\
&= -\pi i.
\end{aligned}$$

Remark. Since the principal logarithm (Log) is not continuous on the *negative* real axis, where γ crosses, we cannot use this as the "correct" antiderivative of z^{-1}. Indeed, using Log yields the incorrect answer πi.

Remark. One needs not necessarily use a branch of log. For example, the function $g(z) = \text{Log}(-z)$ is an antiderivative of $1/z$, and g is analytic in $\mathbb{C} \setminus [0, \infty)$. Thus, we could compute

$$\int_\gamma \frac{1}{z} dz = g(i) - g(i) = \text{Log}(-i) - \text{Log}(i) = -\pi i.$$

Note that $g(z) = L(z) - \pi i$.

Remark. Finally, this could also be computed using Cauchy's theorem. Let γ_0 be the negatively-oriented unit semicircle from $-i$ to i, i.e.,

$$\gamma_0(t) = e^{-it}, \qquad \frac{\pi}{2} \leq t \leq \frac{3\pi}{2}.$$

An explicit calculation using this parametrization yields

$$\int_{\gamma_0} \frac{1}{z} dz = -\pi i.$$

Now, let γ be as before, and consider the curve $\gamma - \gamma_0$ (i.e., traverse γ as usual, and then close the curve by traversing backwards along γ_0). Since $D = \mathbb{C} \setminus [0, \infty)$ is simply connected, and since $g(z) = 1/z$ is analytic on D, then Cauchy's theorem implies

$$0 = \int_{\gamma - \gamma_0} \frac{1}{z} dz = \int_{\gamma} \frac{1}{z} dz - \int_{\gamma_0} \frac{1}{z} dz = \int_{\gamma} \frac{1}{z} dz + \pi i.$$