MAT 334 Test 2 Solutions

Problem 1. *Consider the following power series about $z_0 = 0$:*

$$\sum_{n=0}^{\infty}(2 \cdot 3^n \cdot z^n).$$

What is its radius of convergence?

Solution. Several methods of determining the radius of convergence have been discussed in class. Basically, all these methods will work for this problem.

- *Root test:* The coefficients of the series is $a_n = 2 \cdot 3^n$. Since

  $$\lim_n |a_n|^{\frac{1}{n}} = \lim_n (2^{\frac{1}{n}} 3) = 3,$$

  then by the root test, the radius of convergence is 1/3.

- *Ratio test:* With the a_n's as above, since

  $$\lim_n \frac{|a_{n+1}|}{|a_n|} = \lim_n 3 = 3,$$

  then by the ratio test, the radius of convergence is 1/3.

- *Geometric series:* Since $\sum_n w^n$ converges if and only if $|w| < 1$, then

  $$\sum_{n=0}^{\infty}(2 \cdot 3^n \cdot z^n) = 2\sum_{n=0}^{\infty}(3z)^n$$

  converges if and only if $|3z| < 1$, that is, if and only if $|z| < 1/3$. It follows that the radius of convergence is 1/3.

Problem 2. *Find the entire function $f = u + iv$ such that:*

- $f(0) = 2$
- *The imaginary part of f satisfies $v(x, y) = x$.*

State your final answer for f in terms of z, not x and y.

Solution. It is not too hard to determine what f is by just staring at the problem hard enough, but it can be computed in a straightforward manner using the Cauchy-Riemann equations. In particular, for f to be analytic, u and v must satisfy the Cauchy-Riemann equations,

$$\partial_x u(x,y) = \partial_y v(x,y) = 0, \qquad \partial_y u(x,y) = -\partial_x v(x,y) = -1.$$

Integrating the second Cauchy-Riemann equation in y yields

$$u(x,y) = -y + g(x),$$

for some function g of x. Using the first Cauchy-Riemann equation, we have

$$g'(x) = \partial_x u(x,y) = 0,$$

hence g is a constant function. Combining the above, we obtain

$$u(x,y) = -y + C$$

for some constant C, and hence

$$f(x,y) = u(x,y) + iv(x,y) = (-y + ix) + C = i(x + iy) + C.$$

Finally, the condition $f(0) = 2$ implies that $C = 2$. Since $z = x + iy$, then the final answer can be stated as $f(z) = iz + 2$.

Problem 3. *Let γ denote the positively oriented boundary of a triangle with vertices at $-1-i$, $1-i$, and i. Evaluate*

$$\int_\gamma \frac{dz}{z^2 + 2iz}.$$

Solution. Since γ is a piecewise smooth closed curve, we hope to apply Cauchy's formula to the above contour integral.

First of all, the integrand can be factored as

$$\frac{1}{z^2 + 2iz} = \frac{1}{z(z+2i)},$$

which is singular at two points, $z = 0$ and $z = -2i$. Note that $z = 0$ is inside γ, while $z = -2i$ is outside. Thus, by letting D be some open simply connected region containing γ and its interior, but not containing $-2i$, then

$$f : D \to \mathbb{C}, \qquad f(z) = \frac{1}{z + 2i}$$

is analytic on D. Thus, by Cauchy's formula, we can compute

$$\int_\gamma \frac{dz}{z^2+2iz} = \int_\gamma \frac{f(z)}{z}dz = 2\pi i \cdot n(\gamma;0)\cdot f(0) = 2\pi i \cdot 1 \cdot \frac{1}{2i} = \pi.$$

Problem 4. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire, and assume that $|f(z)| > 1$ for all $z \in \mathbb{C}$. What can you say about f?*

Solution. Let $g : \mathbb{C} \to \mathbb{C}$ be the function $g(z) = 1/f(z)$ (note that since $|f(z)| > 1$ for all $z \in \mathbb{C}$, then $g(z)$ is well-defined for all $z \in \mathbb{C}$). Moreover,

$$|g(z)| = \frac{1}{|f(z)|} < 1, \qquad z \in \mathbb{C}.$$

In particular, g is an entire function which is everywhere bounded. By Liouville's theorem, g is a constant function, hence $f = 1/g$ is also constant,

$$f(z) = \lambda, \qquad z \in \mathbb{C}.$$

Finally, since $|f| > 1$ everywhere, the constant λ must satisfy $|\lambda| > 1$.

Problem 5. *Recall that the* **index** *of a curve γ with respect to $z_0 \in \mathbb{C}$ is*

$$n(\gamma; z_0) = \frac{1}{2\pi i}\int\limits_\gamma \frac{dz}{z - z_0}.$$

Evaluate $n(\gamma;0)$, where γ is the following piecewise smooth, closed curve:

$$\gamma : [0,6] \to \mathbb{C}, \qquad \gamma(t) = \begin{cases} (1+t)e^{2\pi i t} & 0 \le t \le 5, \\ 6 - 5(t-5) & 5 \le t \le 6. \end{cases}$$

Solution. The index $n(\gamma;0)$ is precisely the number of times γ goes around 0 counterclockwise. By graphing γ, you can (or should be able to) see that γ spirals around 0 counterclockwise five times. Thus, $n(\gamma;0) = 5$.