Spring 2014 MAT 336 Practice Exam 1

Here are full solutions to the practice exam questions, along with some ideas on how to think about them. There is a bit more detail in these write-ups than would be expected of you in a heavily timed exam.

Problem 1. *Does the series*

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^4+4}}$$

converge? Justify your answer.

Idea. The simplest way to do this is to use the comparison test to reduce our problem to checking the convergence of a much easier series.

Solution. Since for each n,

$$\sqrt{n^4+4} \geq \sqrt{n^4} = n^2,$$

then for each n, we have that

$$\frac{1}{\sqrt{n^4+4}} \leq \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} n^{-2}$ converges (see, e.g., your last homework assignment), then the comparison test implies that the series in question also converges.

Problem 2. *Find* $\liminf_n x_n$*, where*

$$x_n = \left(2 + \frac{1}{n}\right) \cos \frac{\pi n}{6}.$$

Idea. Note first that $2 + n^{-1}$ becomes arbitrarily close to 2, while $\cos(\pi n/6)$ oscillates along several values between -1 and 1. Thus, the values of x_n are minimized when the cosine factor becomes -1 (which happens infinitely many times). Thus, we expect $\liminf_n x_n$ to be $2 \cdot (-1) = -2$.

Solution. First, we need only consider what happens for large n, for which $2 + n^{-1}$ is very close to 2. For such n, the infimum of the corresponding tail,

$$m_n = \inf\{x_n, x_{n+1}, x_{n+2}, \dots\},$$

is achieved by x_{k_n}, where k_n is the smallest number such that $k_n \geq n$ and $\cos(\pi k_n/6) = -1$ (i.e., $k_n/6$ is an odd integer). Thus,

$$m_n = x_{k_n} = -(2 + k_n^{-1}).$$

As a result, since k_n increases to ∞ as $n \to \infty$, then

$$\liminf_n x_n = \lim_n m_n = -\lim_n (2 + k_n^{-1}) = -2.$$

Problem 3. *Let $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be defined by*

$$d(x, y) = (x - y)^2.$$

Is d a metric on $\mathbb{R}$? Why or why not?

Idea. The thing to do is to test each condition in the definition the metric. Either all conditions are satisfied, so that d is indeed a metric, or some condition is violated, so that d is not a metric.

Solution. d does not satisfy the triange inequality—for example,

$$d(2, 0) = (2 - 0)^2 = 4, \quad d(1, 0) = (1 - 0)^2 = 1, \quad d(2, 1) = (2 - 1)^2 = 1,$$

hence it follows that

$$d(2, 0) = 4 \not\leq 2 = d(2, 1) + d(1, 0).$$

Thus, d is not a metric on $\mathbb{R}$.

Problem 4. *Show that the set*

$$(-1, 0) \cup (0, 1) = \{x \in \mathbb{R} \mid -1 < x < 0 \text{ or } 0 < x < 1\}$$

has the same cardinality as $\mathbb{R}$.

Idea. While it is possible to directly construct a one-to-one correspondence between $(-1, 0) \cup (0, 1)$ and $\mathbb{R}$, it is far, far easier to construct two injections and use the Schröder-Bernstein theorem. We do this below.

Solution. The identity function

$$f_1 : (-1, 0) \cup (0, 1) \to \mathbb{R}, \qquad f_1(x) = x$$

is an injection (a one-to-one function), so that

$$|(-1,0) \cup (0,1)| \leq |\mathbb{R}|.$$

For the other direction, we note that

$$f_2 : \mathbb{R} \to (0,1), \qquad f_2(x) = \frac{1}{\pi}\left(\operatorname{Tan}^{-1} x + \frac{\pi}{2}\right)$$

is in fact a one-to-one correspondence between $\mathbb{R}$ and $(0,1)$. Thus,

$$|\mathbb{R}| \leq |(-1,0) \cup (0,1)|.$$

From these two comparisons, we obtain, via the Schröder-Bernstein theorem,

$$|(-1,0) \cup (0,1)| = |\mathbb{R}|.$$

Problem 5. *Let* (a_n) *be a sequence of real numbers, and suppose* $\lim_n a_n = L$. *Show the following limit also holds:*

$$\lim_n \frac{1}{n}(a_1 + a_2 + \cdots + a_n) = L.$$

Idea. Note the quantity inside the limit is just the average of all the a_k's up to n. While anything can happen for small k's, in contrast, a_k will be very close to L for large k's. Thus, as n becomes larger and larger, then more and more weight in this average will be on a_k's that are very close to L. As $n \to \infty$, the contribution of the early a_k's (which can be far from L) will tend to zero, so that the average will be as close to L as one wants.

Solution. Fix $\varepsilon > 0$. Since $a_n \to L$, there exists N_1 such that

$$|a_n - L| < \frac{\varepsilon}{2}, \qquad n \geq N_1.$$

With N_1 fixed, we can now find N_2 such that

$$\frac{1}{n}\sum_{k=1}^{N_1} |a_k - L| < \frac{\varepsilon}{2}, \qquad n \geq N_2.$$

Thus, as long as $n \geq N = \max(N_1, N_2)$, we can bound

$$\left|\frac{1}{n}(a_1 + \ldots a_n) - L\right| \leq \frac{1}{n}\sum_{k=1}^{n} |a_k - L|$$

$$
\begin{aligned}
&\leq \frac{1}{n}\sum_{k=1}^{N_1}|a_k - L| + \frac{1}{n}\sum_{k=N_1+1}^{n}|a_k - L| \\
&< \frac{\varepsilon}{2} + \frac{1}{n}\sum_{k=N_1+1}^{n}\frac{\varepsilon}{2} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}
$$

It follows that $n^{-1}(a_1 + \ldots a_n) \to L$, as desired.

Extra Problem. *Give an example of a sequence (x_n) in $\mathbb{R}$ such that*

$$\lim_n |x_{n+1} - x_n| = 0,$$

but (x_n) does not converge to a real number.

Idea. We want to find a sequence (x_n) such that consecutive points are as close together as we want, but the points of the sequence still "escape" and do not clump anywhere. This can be done by simply experimenting.

For a bit more insight on how to construct such a sequence, though, consider the following expression of x_n as a telescoping sum:

$$x_n = x_0 + \sum_{k=1}^{n}(x_k - x_{k-1}).$$

Thus, we see the convergence of (x_n) is equivalent to that of the infinite series $\sum_{k=1}^{\infty} a_k$, where $a_k = x_k - x_{k-1}$. From this perspective, it seems reasonable to find a_k such that $a_k \to 0$, but $\sum_{k=1}^{\infty} a_k$ diverges. One example of this is the harmonic series, $a_k = 1/k$, on which we base our answer below.

Solution. Consider the sequence

$$x_n = \sum_{k=1}^{n}\frac{1}{k}.$$

Since these are the partial sums for the harmonic series, (x_n) does not converge to a real number. However,

$$\lim_n |x_{n+1} - x_n| = \lim_n \frac{1}{n+1} = 0.$$