Spring 2014 MAT 336 Practice Exam 2

Here are solutions to the practice exam questions.

Problem 1. *Show that* $\mathbb{Q}$ *is not a connected subset of* $\mathbb{R}$*.*

Solution. Consider any irrational number $x \in \mathbb{R} \setminus \mathbb{Q}$. Then, $\mathbb{Q}$ can be written as the union of two disjoint open sets,

$$\mathbb{Q} = (-\infty, x) \cup (x, \infty),$$

hence $\mathbb{Q}$ is disconnected.

Problem 2. *Suppose* U *and* V *are open subsets of* $\mathbb{R}$*. Show that*

$$U \times V = \{(x, y) \in \mathbb{R}^2 \mid x \in U,\ y \in V\}$$

is an open subset of $\mathbb{R}^2$*.*

Solution. Suppose $(x, y) \in U \times V$. Since $x \in U$, there is some $r_x > 0$ such that $B_{\mathbb{R}}(x, r_x) \subseteq U$. Similarly, since $y \in V$, there is some $r_y > 0$ such that $B_{\mathbb{R}}(y, r_y) \subseteq V$. Now, if $r = \min(r_x, r_y)$, and if $(x', y') \in B_{\mathbb{R}^2}((x, y), r)$, then

$$d_{\mathbb{R}}(x', x) < r \leq r_x, \qquad d_{\mathbb{R}}(y', y) < r = r_y,$$

so that $x' \in U$, $y' \in V$, and hence $(x', y') \in U \times V$. As a result, we have $B_{\mathbb{R}^2}((x, y), r) \subseteq U \times V$, and it follows that $U \times V$ is indeed open.

Problem 3. *Suppose* $f : (-1, 1) \to \mathbb{R}$ *is differentiable at* 0*, and suppose* $f'(0) > 0$*. Show that there is some* $x \in (0, 1)$ *such that* $f(x) > f(0)$*.*

Solution. Since

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = f'(0) > 0,$$

there is some $0 < \delta < 1$ such that

$$\left| \frac{f(h) - f(0)}{h} - f'(0) \right| < f'(0)$$

for all $0 < |h| < \delta$. In particular, for such h, we have

$$\frac{f(h) - f(0)}{h} = \left[\frac{f(h) - f(0)}{h} - f'(0) \right] + f'(0) > 0.$$

Finally, taking a positive $0 < h < \delta$ and rearranging the above inequality, we obtain that $f(h) > f(0)$, as desired.

Problem 4. *Show that* $1 - \cos x \leq x \sin x$ *for all* $x \in [0, \frac{\pi}{2}]$. Hint: Apply the mean value theorem, and note that $\sin x$ is monotone on $[0, \frac{\pi}{2}]$.

Solution. Applying the mean value theorem to $f(x) = \cos x$, we obtain

$$\cos x - \cos 0 = \cos' x' \cdot (x - 0),$$

for some $x' \in [0, x]$. This can be simplified as

$$1 - \cos x = \sin x' \cdot x.$$

Since sin is nondecreasing on $[0, \frac{\pi}{2}]$, then $\sin x' \leq \sin x$, and hence

$$1 - \cos x \leq x \sin x.$$

Problem 5. *Suppose* $f : [0, 1] \to [0, 1]$ *is continuous. Show that* f *has a fixed point, i.e., there is some* $x \in [0, 1]$ *such that* $f(x) = x$.

Solution. Let $g : [0, 1] \to \mathbb{R}$ be defined by $g(x) = f(x) - x$. Then, to solve our problem, it suffices to find $x \in [0, 1]$ satisfying $g(x) = 0$.

Note that g is continuous, $g(0) = f(0) \geq 0$, and $g(1) = f(1) - 1 \leq 0$. If either $g(0)$ or $g(1)$ is zero, then we are trivially done. Otherwise, we have both $g(0) > 0$ and $g(1) < 0$, hence the intermediate value theorem implies that there exists some $x \in (0, 1)$ such that $g(x) = 0$.