Spring 2014 MAT 336 Practice Exam 3

Answer questions! Get points!

Problem 1. *Consider the sequence of functions*

$$f_n : [0, 2\pi] \to \mathbb{R}, \qquad f_n(x) = \sin(nx).$$

Does (f_n) *converge uniformly to some* $g : [0, 2\pi] \to \mathbb{R}$*? Why or why not?*

Problem 2. *Let* (X, d) *be a metric space, and let* K_1 *and* K_2 *be compact subsets of* X*. Show that* $K_1 \cup K_2$ *is also a compact subset of* $\mathbb{R}^n$*.*

Problem 3. *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is differentiable, and suppose* $|f'(x)| \leq 2$ *for any* $x \in \mathbb{R}$*. Show that* f *is Lipschitz, with Lipschitz constant* 2*, i.e.,*

$$|f(x) - f(y)| \leq 2|x - y|, \qquad x, y \in \mathbb{R}.$$

Problem 4. *Consider the space* X *of all continuous maps* $f : [-1, 1] \to \mathbb{R}$ *which are also differentiable on* $(-1, 1)$*, with metric*

$$d : X \times X \to [0, \infty), \qquad d(f, g) = \sup_{x \in [-1,1]} |f(x) - g(x)|.$$

Show that (X, d) *is not complete.*

Problem 5. *Suppose* $f : [-1, 0] \to \mathbb{R}$ *and* $g : [0, 1] \to \mathbb{R}$ *are (Riemann) integrable, and suppose* $f(0) = g(0)$*. Define now*

$$h : [-1, 1] \to \mathbb{R}, \qquad h(x) = \begin{cases} f(x) & -1 \leq x \leq 0, \\ g(x) & 0 \leq x \leq 1. \end{cases}$$

Show that h *is integrable. What is* $\int_{-1}^{1} h(x)dx$*?*