Spring 2014 MAT 336 Exam 1

These are the fully detailed solutions to all the questions on Midterm 1, along with some remarks on how one can approach the problems.

Problem 1. *Consider the set*

$$A = \{(1 + x^2)^{-1} \mid x \in \mathbb{R}\}.$$

Find $\sup A$ *and* $\inf A$*, and justify your answers.*

Idea. If you understand the definitions of sup and inf, it should be clear what the answers are, as long as you can see that $(1 + x^2)^{-1}$ takes values on the interval $(0, 1]$. All that is left to do is to justify your answer.

Solution. Since $(1 + x^2) > 0$ for any $x \in \mathbb{R}$, then

$$\frac{1}{1 + x^2} > 0, \qquad x \in \mathbb{R},$$

so 0 is a lower bound for A. Similarly, since $1 + x^2 \geq 1$ for any x, then

$$\frac{1}{1 + x^2} \leq 1, \qquad x \in \mathbb{R},$$

and it follows that 1 is an upper bound for A.

We claim 0 is the *greatest* lower bound, and 1 is the *least* upper bound:

- Since $1 = (1 + 0^2)^{-1} \in A$, no $y < 1$ can be an upper bound of A.
- If $0 < y < 1$, then choosing $x \in \mathbb{R}$ such that $x^2 > y^{-1} - 1 > 0$, we have

  $$\frac{1}{1 + x^2} < \frac{1}{1 + (y^{-1} - 1)} = y.$$

  Thus, y cannot be a lower bound of A.

From the above, we see that $\sup A = 1$ and $\inf A = 0$.

Problem 2. *Find the following limit:*

$$\lim_n [\log(n + 3) - \log n].$$

Prove that your answer is correct.

Remark. Yes, it is true that log is continuous, and hence

$$\lim_n [\log(n+3) - \log n] = \lim_n \log \frac{n+3}{n} = \log \lim_n \frac{n+3}{n} = \log 1 = 0.$$

(You will get a fair share of points if you said the above, with some justifications). But, we have not covered continuity yet in this course, and the point here is to justify this answer 0 using the tools we have developed.

Idea. To find how to set up the ε-N-argument correctly, we work backwards. From the above, we already see that the limit should be zero. Thus, for a fixed ε, we need to find n such that

$$|\log(n+3) - \log n - 0| = \log \frac{n+3}{n} < \varepsilon.$$

From here, this is just a bit of algebra:

$$1 + \frac{3}{n} < e^{\varepsilon}, \qquad \frac{3}{n} < e^{\varepsilon} - 1, \qquad n > \frac{3}{e^{\varepsilon} - 1}.$$

Thus, the ε-N-argument should go through for any $n > 3(e^{\varepsilon} - 1)^{-1}$.

Solution. We wish to show the limit is 0. Fix $\varepsilon > 0$. Choosing N large enough so that $N > 3(e^{\varepsilon} - 1)^{-1}$, then for any $n \geq N$, we have that

$$\frac{3}{n} \leq \frac{3}{N} < e^{\varepsilon} - 1.$$

As a result, when $n \geq N$,

$$|[\log(n+3) - \log n] - 0| = \log\left(1 + \frac{3}{n}\right) < \log[1 + (e^{\varepsilon} - 1)] = \varepsilon.$$

Thus, we have shown that the desired limit is 0.

Problem 3. *Give a specific example of a sequence (x_n) in $\mathbb{R}$ and a subsequence (x_{n_k}) of (x_n), such that $\sum_{n=0}^{\infty} x_n$ converges, but $\sum_{k=0}^{\infty} x_{n_k}$ diverges.*

Idea. At worst, you can fumble around in the dark until you bump into a sequence and subsequence that works. However, if you recall some things we learned about series, you can limit your search possibilities.

In particular, if the x_n's are all nonnegative, and if $\sum_{n=0}^{\infty} x_n$ converges, then the summation over any subsequence will also converge, either by the

Cauchy criterion or by the comparison test. Thus, we will need a series with both positive and negative terms. This further narrows your search.

Furthermore, if you do not want to search around at all, you can recall the discussion we had about series $\sum_{n=0}^{\infty} x_n$ which converge but do not converge absolutely. In particular, in the proof that such series can be rearranged to produce any limit, we mentioned that if we considered the subsequence (x_{n_k}) of all positive or negative x_n's, then the sums over these x_{n_k}'s must diverge. The simplest example of this is the alternating harmonic series, $\sum_{n=1}^{\infty}(-1)^n n^{-1}$, which we use for our solution below.

Solution. Let $x_n = 2(-1)^n n^{-1}$ (you can take $x_0 = 0$ if you want to cover all loose ends). Since this is an alternating series, with $|x_n|$ monotone decreasing, then $\sum_{n=1}^{\infty} x_n$ converges. However, the subsequence (x_{2k}) satisfies

$$\sum_{k=1}^{\infty} x_{2k} = \sum_{k=1}^{\infty} \frac{2}{2k} = \sum_{k=1}^{\infty} \frac{1}{k},$$

which diverges, as it is simply the harmonic series.

Problem 4. *Let d be the metric on $X = [1, \infty)$ given by*

$$d(x, y) = \left| \frac{1}{x} - \frac{1}{y} \right|.$$

(You do NOT have to show d is a metric.) *Show that (X, d) is* not *complete.*

Idea. The first point is that you have to unwind the definition of completeness: you have to find a Cauchy sequence on $[1, \infty)$ (with respect to d) that does not converge. Now, since $d(x, y)$ is defined to be $|x^{-1} - y^{-1}|$ rather than $|x - y|$, this has the effect of massively shrinking distances between points near infinity. Thus, a viable strategy is to take a sequence (x_n) tending toward $+\infty$. Such a sequence cannot converge ($+\infty$ is not in our space), but may be Cauchy under our new metric d.

In fact, it turns out that any sequence going to $+\infty$ will work!

Solution. Consider $x_n = n$, for $n \geq 1$. We claim (x_n) is a Cauchy sequence.

Fix $\varepsilon > 0$. Choosing N so that $N > \varepsilon^{-1}$, we see that for any $n > m \geq N$,

$$d(x_n, x_m) = \left| \frac{1}{n} - \frac{1}{m} \right| = \frac{n - m}{nm} \leq \frac{1}{m} \leq \frac{1}{N} < \varepsilon.$$

(In the above, we used that $(n-m)/n \leq 1$.) This proves (x_n) is Cauchy.

Furthermore, (x_n) cannot converge to any limit (this probably does not require a proof, but we do it here anyway). Given $y \in [1, \infty)$, if $n \geq 2y$, then

$$d(x_n, y) \geq (y^{-1} - (2y)^{-1}) = (2y)^{-1}.$$

Thus, (x_n) cannot converge to y.

Since we have constructed a Cauchy sequence (x_n) in (X, d) that does not converge to a limit, then (X, d) is not complete.

Problem 5. *Let (x_n) be a bounded sequence in $\mathbb{R}$, and let $\ell = \limsup_n x_n$. Show that there exists a subsequence (x_{n_k}) that converges to ℓ.*

Idea. This is a problem in which you have to unwind definitions, in particular for lim sup. It's a rather typical proof involving lim sup, but if you have little experience with such proofs, it can be a very tough problem.

The basic strategy is as follows: recall $\limsup_n x_n = \lim_n M_n$, where

$$M_n = \sup\{x_k \mid k \geq n\}$$

is the supremum of the tail of (x_n) starting from n. The goal is then to pick the x_n's that come close enough to achieving these suprema M_n.

Solution. Let $x_{n_0} = x_0$, that is, $n_0 = 0$. Given x_{n_k}, we consider

$$\mathcal{M}_k = M_{n_k+1} = \sup\{x_l \mid l > n_k\}.$$

Since $\mathcal{M}_k$ is the *least* upper bound of these x_l's, then we can find some $n_{k+1} > n_k$ satisfying $\mathcal{M}_k - 2^{-k} < x_{n_{k+1}} \leq \mathcal{M}_k$. This process produces a subsequence $x_{n_0} = x_0, x_{n_1}, x_{n_2}, x_{n_3}, \dots$ of (x_n).

Since $n_k \to \infty$ as $k \to \infty$, then by definition,

$$\lim_k \mathcal{M}_k = \limsup_n x_n = \lim_k (\mathcal{M}_k - 2^{-k}).$$

Thus, by the squeeze theorem, we obtain

$$\lim_k x_{n_k} = \limsup_n x_n.$$