Spring 2014 MAT 336 Exam 2

These are the solutions to all the questions on Midterm 2.

Problem 1. *Give an example of a continuous function* $f : \mathbb{R}^2 \to \mathbb{R}$ *and an open subset* $U \subseteq \mathbb{R}^2$ *such that* $f(U) = \{f(x) \mid x \in U\}$ *is not open.*

Remark. There are many, many examples. Below, we run with the simplest.

Solution. Let $f(x) = 0$, and let $U = \mathbb{R}^2$ (which is open). The $f(U) = \{0\}$, which is not open (any open ball centered at 0 contains more than just 0).

Problem 2. *Show that* $\ln(1 + x) \leq x$ *for any* $x \geq 0$*.* Hint: One possibility is to find a way to apply the mean value theorem.

Solution. We apply the mean value theorem to $f(x) = \ln(1 + x)$, which is differentiable for all $x \in (-1, \infty)$. In particular, applying at $x_0 = 0$, we have

$$\ln(1 + x) = f(x) - f(0) = f'(x') \cdot (x - 0),$$

for some $x' \in [0, x]$. Noting that

$$f'(x') = \frac{1}{1 + x'} \in (0, 1],$$

we see that

$$\ln(1 + x) = \frac{1}{1 + x'} \cdot x \leq x.$$

Problem 3. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be continuous, and suppose* f *is periodic, that is, there is some* $d > 0$ *such that* $f(x + d) = f(x)$ *for all* $x \in \mathbb{R}$*. Show that* f *achieves its maximum value (more specifically, show there is some* $x_0 \in \mathbb{R}$ *such that* $f(x_0) \geq f(x)$ *for all* $x \in \mathbb{R}$*).*

Solution. Consider the restriction of f to $[0, d]$, i.e.,

$$g : [0, d] \to \mathbb{R}, \qquad g(x) = f(x).$$

Since g is continuous and $[0, d]$ is compact (by the Heine-Borel theorem), g achieves its maximum at some $x_0 \in [0, d]$.

Now, for any $x \in \mathbb{R}$, we can write $x = md + x'$, where $m \in \mathbb{Z}$ and $x' \in [0, d)$. Thus, by the periodicity of f, we have that

$$f(x) = f(x') = g(x') \leq g(x_0) = f(x_0).$$

In other words, f also achieves its maximum at x_0 (as well as at $x_0 + md$).

Problem 4. *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *satisfies that*

$$|f(x) - f(y)| \leq (x - y)^2$$

for all $x, y \in \mathbb{R}$. *Prove that* f *is a constant function.*

Solution. Rearranging the above inequality, we see that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq |x - y|, \qquad x, y \in \mathbb{R}, \quad x \neq y.$$

Fixing y and taking limits $x \to y$ on both sides of the inequality yields

$$|f'(y)| \leq 0.$$

Since the above holds for all $y \in \mathbb{R}$, then f' vanishes everywhere. As a result, (by the mean value theorem,) f is everywhere constant.

Problem 5. *Consider the set* $A = \{(x, \sin \frac{1}{x}) \in \mathbb{R}^2 \mid x > 0\} \subseteq \mathbb{R}^2$. *Find all the limit points of* A.

Solution. Clearly, every point $(x, y) \in A$ is a limit point of A (since the constant sequence $(x_n, y_n) = (x, y)$ trivially converges to (x, y)).

In addition, we claim that every point $(0, y)$, where $y \in [-1, 1]$, is a limit point of A. Since sin is periodic, there is a sequence x_n going to $+\infty$, with $\sin x_n = y$. Then, $(x_n^{-1}, \sin x_n)$ is a sequence in A which converges to $(0, y)$.

In fact, the set of limit points of A is precisely

$$\bar{A} = A \cup \{(0, y) \mid y \in [-1, 1]\}.$$

Remark. If you're feeling extra ambitious, you can also show that any point not in $\bar{A}$, as defined above, is *not* a limit point of A. For example, since for any sequence $(x_n, y_n) \in A$, we have that $x_n > 0$ and that $|y_n| \leq 1$, it follows that (x, y) cannot be a limit point of A if either $x < 0$ or if $|y| > 1$.

Furthermore, for points $(x, y) \notin A$ satisfying $x > 0$, we can use, for example, Exercise 5.3.I in the textbook (which was a homework problem) to find a ball $B((x, y), r)$ that avoids A. As a result, since (x, y) is of a positive distance from A, no sequence in A can approach (x, y).

The above cases combined cover all points not in $\bar{A}$.