SOLUTIONS OF SELECTED EXERCISES IN T. TAO'S *NONLINEAR DISPERSIVE EQUATIONS*

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This is a list of solutions to some of the exercises in the book *Nonlinear Dispersive Equations: Local and Global Analysis*, by T. Tao. [1] Many of the later problems (beginning from Section 2.3) were done in collaboration with the *Nonlinear Dispersive Equations* reading group (Jordan Bell, David Reiss, Kyle Thompson) at the University of Toronto.

Chapter 1: Ordinary Differential Equations

1.2. First of all, fixed points are unique, since if $u, v \in X$ are fixed points of Φ, then

$$d(u, v) = d(\Phi(u), \Phi(v)) \leq c d(u, v),$$

which is only possible when $d(u, v) = 0$, i.e., $u = v$.

Next, fix any $u_0 \in X$, and define recursively $u_{k+1} = \Phi u_k$. By induction and the contraction mapping property, we have $d(u_k, u_{k+1}) \leq c^k d(u_0, u_1)$, and hence for any $m \leq n$,

$$d(u_m, u_n) \leq \sum_{k=m}^{n-1} d(u_k, u_{k+1}) \leq d(u_0, u_1) \sum_{k=m}^{n-1} c^k \leq \frac{c^m d(u_0, u_1)}{1 - c}.$$

In particular, $\{u_k\}$ is a Cauchy sequence, so there is some $u \in X$ such that $u_k \to u$. Since contraction mappings are clearly continuous (by the contraction property), then

$$\Phi(u) = \lim_k \Phi(u_k) = \lim_k u_{k+1} = u,$$

and hence u is the fixed point of Φ.

Finally, for any $v \in X$, we define $v_0 = v$ and $v_{k+1} = \Phi v_k$, as before. By continuity,

$$d(v, u) = \lim_k d(v_0, v_k) \leq \lim_k \sum_{i=0}^{k-1} d(v_i, v_{i+1}) \leq d(v_0, v_1) \sum_i c^i = \frac{1}{1 - c} d(v, \Phi(v)).$$

1.3. [2] Let $A = \nabla\Phi(x_0)$. Since A is nonsingular by assumption, there exists $\lambda > 0$ such that $2\lambda \|A^{-1}\| \leq 1$, with $\|\cdot\|$ denoting the operator norm.

Fix $y \in \mathcal{D}$, and define the map $\varphi_y : \mathcal{D} \to \mathcal{D}$ by

$$\varphi_y(x) = x + A^{-1}[y - \Phi(x)].$$

Note that x is a fixed point of φ_y if and only if $\Phi(x) = y$. Taking the differential, we obtain

$$\nabla\varphi_y(x) = I - A^{-1}\nabla\Phi(x) = A^{-1}[A - \nabla\Phi(x)].$$

By continuity, there exists a neighborhood U of x_0 such that $\|A - \nabla\Phi(x)\| < \lambda$ for all $x \in U$. Consequently, for any $x \in U$, we have the bound

$$\|\nabla\varphi_y(x)\| \leq \|A^{-1}\| \|A - \nabla\Phi(x)\| < \frac{1}{2}.$$

[1] See [5].

[2] The solution was obtained mostly from [3].

It follows that φ_y is Lipschitz on U, with Lipschitz constant less than $1/2$, i.e.,

$$|\varphi_y(x_1) - \varphi_y(x_2)| \leq \frac{1}{2}|x_1 - x_2|, \qquad x_1, x_2 \in U.$$

In particular, if $\Phi(x_1) = \Phi(x_2)$, then $\varphi_y(x_1) - \varphi_y(x_2) = x_1 - x_2$, and hence $2|x_1 - x_2| \leq |x_1 - x_2|$, i.e., $x_1 = x_2$. From this, we conclude that Φ is one-to-one on U.

Next, fix any $x_1 \in U$, let $V = \Phi(U)$, and let $y_1 = \Phi(x_1) \in V$. Choose $r > 0$ such that $\bar{B} = \overline{B(x_1, r)}$ is contained in U. [3] If $y \in \mathcal{D}$ is such that $|y - y_1| < \lambda r$, then for any $x \in \bar{B}$,

$$|\varphi_y(x) - x_1| \leq |\varphi_y(x) - \varphi_y(x_1)| + |\varphi_y(x_1) - x_1| \leq \frac{1}{2}|x - x_0| + \|A^{-1}\||y - y_0| \leq r.$$

Therefore, φ_y maps from $\bar{B}$ into $\bar{B}$. In particular, $\varphi_y|\bar{B}$ is a contraction mapping on the complete metric space $\bar{B}$, so that φ_y has a unique fixed point $z \in \bar{B}$. This implies that $\Phi(z) = y$, so that $y \in V$. With this, we have now proved that V is open, and that Φ is a one-to-one mapping from U onto V.

Let $\Psi : V \to U$ be the inverse of Φ. Let $y, y + k \in V$, and define

$$x = \Psi(y), \qquad x + h = \Psi(y + k).$$

Since

$$|h - A^{-1}k| = |h + A^{-1}[\Phi(x) - \Phi(x + h)]| = |\varphi_y(x + h) - \varphi_y(x)| \leq \frac{1}{2}|h|,$$

it follows that $|h| \leq 2|A^{-1}k| \leq \lambda^{-1}|k|$. Moreover, since

$$\|I - A^{-1}\nabla\Phi(x)\| \leq \|A^{-1}[A - \nabla\Phi(x)]\| \leq \frac{1}{2} < 1$$

by all our previous assumptions, then $A^{-1}\nabla\Phi(x)$, and hence $\nabla\Phi(x)$, is invertible.

Let $S = \nabla\Phi(x)$, and let $T = S^{-1}$. A direct computation yields

$$\begin{aligned} \frac{|\Psi(y + k) - \Psi(y) - Tk|}{|k|} &= \frac{|-T[\Phi(x + h) - \Phi(x) - Sh]|}{|k|} \\ &\leq \frac{\|T\|}{\lambda} \cdot \frac{|\Phi(x + h) - \Phi(x) - Sh|}{|h|}. \end{aligned}$$

The right-hand side goes to zero as $|h| \searrow 0$. Since we have proved this for arbitrary y and $y + k \in V$, then Ψ is differentiable on V, and $\nabla\Psi(y) = [\nabla\Phi(\Psi(y))]^{-1}$. Since both Ψ and $\nabla\Phi$ are continuous, the above formula implies that $\nabla\Psi$ is continuous, so that $\Psi = \Phi^{-1}$ is C^1.

1.4. We begin by generating a solution $u \in C^1(I \to \mathcal{D})$ using Theorem 1.7, and we proceed by induction. Suppose u is C^l for $l \leq k$. Then, $\partial_t u = F \circ u$ is C^l as well, which implies that u is C^{l+1}. This iterative process continues until $l = k$, so that u is C^{k+1}. By definition, then $u \in C^{k+1}_{loc}(I \to \mathcal{D})$, and the map $S_{t_0}(t)$ is k times differentiable.

1.5. The Picard existence theorem generalizes directly to higher-order quasilinear ODE, since these can be reformulated equivalently as first-order systems. Similarly, the Picard existence theorem (and also the Cauchy-Kowalevski theorem) extends to non-autonomous systems, since these can be equivalently formulated as autonomous systems.

[3] We let $B(x_0, r)$ denote the open ball of radius r about x_0.

1.6. The infinite iteration scheme described in the problem statement can collapse, since the time intervals $\Delta t_i = t_i - t_{i-1}$, $i \geq 1$, in each iteration step can become arbitrarily small, depending on the growth of the solution at each step. For example, if

$$\sum_{i=1}^{\infty} \Delta t_i < \infty,$$

then we have only a finite-time solution.

In particular, in the case of (1.6), we have $|u(t)| = 1/(1-t)$. Suppose we solve using Picard iteration beginning at time $0 \leq t_0 < 1$. Let Ω, as in the statement of Theorem 1.7, be the ball $B = B(0, 2/(1-t_0))$. In this case, F is given by $F(u) = u^2$, so that

$$\|F\|_{C^0(B)} \leq \frac{4}{(1-t_0)^2}, \qquad \|F\|_{\dot{C}^{0,1}(B)} \leq \frac{4}{1-t_0}.$$

(For the latter inequality, we have $|F(u) - F(v)| \leq |u+v||u-v| \leq (|u| + |v|)|u-v|$.) As a result, we only have local existence on a time interval $T \ll \min((1-t_0)^2, (1-t_0)) \lesssim 1 - t_0$. In particular, $t_0 + T$ will always be smaller than 1, no matter the choice of t_0, resulting in the qualitative description of the preceding paragraph.

1.7. Assume $u(t_0) \leq v(t_0)$, and define the function

$$f(t) = [\max(0, u(t) - v(t))]^2$$

on I. Then, f is differentiable a.e., and when exists,

$$\partial_t f(t) = \begin{cases} 0 & v(t) \geq u(t), \\ 2(u(t) - v(t))(u'(t) - v'(t)) & v(t) \leq u(t). \end{cases}$$

Since I is compact and F is Lipschitz, then when $v(t) \leq u(t)$, we have

$$|\partial_t f(t)| \leq 2|u(t) - v(t)||F(t, u(t)) - F(t, v(t))| \lesssim |u(t) - v(t)|^2.$$

This implies that $|\partial_t f(t)| \lesssim |f(t)|$ almost everywhere on I, so by Gronwall's inequality, then

$$f(t) \leq f(t_0) \exp[(t - t_0)C]$$

for all $t \in I$ and for some constant $C > 0$. Since $f(t_0) = 0$ by definition, then $f(t) = 0$ for all $t \in I$. In other words, $u(t) \leq v(t)$ for all $t \in I$.

Now, assume $u(t_0) < v(t_0)$, let $\epsilon > 0$ be a small constant such that $u(t_0) + \epsilon \leq v(t_0)$, and define $g(t) = [\max(0, u(t) + \epsilon - v(t))]^2$ on I. Again, by differentiating g, we obtain

$$\partial_t g(t) = \begin{cases} 0 & v(t) \geq u(t) + \epsilon, \\ 2(u(t) + \epsilon - v(t))(u'(t) - v'(t)) & v(t) \leq u(t) + \epsilon. \end{cases}$$

Again recalling the Lipschitz property of F, we obtain for almost every $t \in I$ that

$$\begin{aligned} |\partial_t g(t)| &\lesssim |u(t) + \epsilon - v(t)||F(t, u(t)) - F(t, v(t))| \\ &\lesssim |u(t) + \epsilon - v(t)||u(t) - v(t)| \\ &\lesssim |u(t) + \epsilon - v(t)|^2 + \epsilon|u(t) - \epsilon - v(t)| \\ &\lesssim \epsilon^2 + |u(t) + \epsilon - v(t)|^2, \end{aligned}$$

whenever $u(t) + \epsilon \geq v(t)$. Thus, for some constant $C > 0$, independent of ϵ, we have

$$\partial_t g(t) \leq C g(t) + C\epsilon^2, \qquad \partial_t [e^{-C(t-t_0)} g(t)] \leq C e^{-C(t-t_0)} \epsilon^2,$$

for almost every $t \in I$. Integrating the above yields the inequalities [4]

$$e^{-C(t-t_0)}g(t) \leq g(t_0) + \epsilon^2 \int_{t_0}^t Ce^{-C(s-t_0)}ds = \epsilon^2(1 - e^{-C(t-t_0)}),$$
$$g(t) \leq \epsilon^2(e^{C(t_1-t_0)} - 1).$$

Given $t \in I$, if $u(t)+\epsilon \leq v(t)$, then $u(t) < v(t)$ as desired, so there is nothing left to prove. On the other hand, if $u(t) + \epsilon > v(t)$, then the definition of g and the above yield that

$$[u(t) + \epsilon - v(t)]^2 = g(t)^2 \leq \epsilon^2[e^{C(t_1-t_0)} - 1].$$

By choosing ϵ which is small with respect to $C(t_1 - t_0)$, [5] then the above implies

$$u(t) + \epsilon - v(t) \leq \frac{\epsilon}{2}, \qquad u(t) < u(t) + \frac{\epsilon}{2} \leq v(t).$$

1.8. With the notations of Theorem 1.10, let $[t_0, t_1] = [0, 1]$, let $A = 2$, and define

$$B(t) = -1, \qquad u(t) = 1, \qquad 0 \leq t \leq 1.$$

It is clear from computation that

$$u(t) = 1 \leq 2 - t = A + \int_{t_0}^t B(s)u(s)ds, \qquad 0 \leq t \leq 1.$$

However, we have that

$$A \exp\left(\int_{t_0}^t B(s)ds\right) = 2e^{-t},$$

which is smaller than $u(t) = 1$ for t sufficiently close to 1.

To reconcile this with Theorem 1.12, we observe that within the proof of Theorem 1.10, we obtain the inequality

$$\frac{d}{dt}\left(A + \int_{t_0}^t B(s)u(s)ds\right) \leq B(t)\left(A + \int_{t_0}^t B(s)u(s)\right),$$

which reduces precisely to Theorem 1.12. However, if B is allowed to be negative, then

$$A + \int_{t_0}^t B(s)u(s)ds$$

needs no longer be nonnegative, which violates the hypotheses of Theorem 1.12.

1.10. First, we apply the usual Picard theory to obtain a solution $u : (T_-, T_+) \to \mathcal{D}$ to (1.7) for which the interval of existence is maximal. We wish to show that $T_+ = \infty$ and $T_- = -\infty$. We need only show the former, since the latter then follows from inverting the time variable. [6] Suppose now that $T_+ < \infty$; differentiating $\|u(t)\|^2$, we obtain

$$\partial_t(1 + \|u(t)\|^2) = \langle \partial_t u(t), u(t)\rangle \lesssim \|F(u(t))\|\|u(t)\| \lesssim 1 + \|u(t)\|^2,$$

where in the last step, we applied the linear bound for F, along with Cauchy's inequality. Applying the differential Gronwall inequality, then we have

$$(1 + \|u(t)\|^2) \lesssim e^{(t-t_0)C}(1 + \|u_0\|^2) \leq e^{(T_+-t_0)C}(1 + \|u_0\|^2), \qquad t_0 \leq t \leq T_+,$$

for some constant $C > 0$. This contradicts Theorem 1.17, so that $T_+ = \infty$.

Since solutions to (1.7) are unique due to Theorem 1.14, and since solutions have the time translation invariance property (due to (1.7) being an autonomous system), then the

[4] Note that here, we have implicitly derived a slightly more general form of the Gronwall inequality.

[5] Recall that C and $t_1 - t_0$ are both independent of ϵ.

[6] Defining $v(t) = u(-t)$, then $\partial_t v(t) = -F(v(t))$, and $-F$ satisfies the same bounds hypothesized for F.

solution maps obey the desired time translation invariance $S_{t_0}(t) = S_0(t - t_0)$. [7] That $S_0(0) = \mathrm{id}$ follows immediately by the definition of the solution map. By the uniqueness of solutions (Theorem 1.14) and the above time translation invariance,

$$S_0(t')S_0(t) = S_t(t' - t)S_0(t) = S_0(t').^{8}$$

Finally, we observe that for any $t, t' \in \mathbb{R}$, say with $t < t'$, we have the bound

$$\|u(t') - u(t)\| \leq \int_t^{t'} \|F(u(t))\| \lesssim \int_t^{t'} (1 + \|u(s)\|)ds \leq |t' - t| \left[1 + \sup_{t' \leq s \leq t} \|u(s)\|\right].$$

It follows that the solution map is locally Lipschitz.

1.11. Suppose the solution curve $u : (T_-, T_+) \to \mathcal{D}$ be maximal. Recall that the Picard theory implies $|u(t)| \to \infty$ as $t \nearrow T_+$. Thus, when t nears T_+, and hence $u(t)$ is large, we have that $|F(u(t))| \lesssim |u(t)|^p$, and therefore

$$\partial_t |u(t)|^2 \leq \langle F(u(t)), u(t)\rangle \lesssim |u(t)|^{p+1},$$
$$\frac{2}{1-p}\partial_t |u(t)|^{1-p} \lesssim [|u(t)|^2]^{\frac{-p-1}{2}} \partial_t |u(t)|^2 \lesssim 1.$$

Integrating the above, we obtain

$$|u(t)|^{1-p} = |u(t)|^{1-p} - \lim_{s \nearrow T_+} |u(s)|^{1-p} \lesssim_p \int_t^{T_+} ds = T_+ - t.$$

Taking the above to the $1/(1 - p)$ power yields our desired lower bound for near T_+. The analogous lower bound near T_- can be proved similarly.

To see that this blowup rate is sharp, consider the case $\mathcal{D} = \mathbb{R}$ and the nonlinearity

$$F(u) = (p - 1)^{-1}|u|^{p-1}u.$$

Consider this particular ODE, with initial condition $u(0) = 1$. A simple change of variables yields the relation $\partial_t |u|^{1-p} \equiv -1$, hence it has the explicit solution

$$|u|^{1-p} - 1 = -t, \qquad u(t) = (1 - t)^{\frac{-1}{p-1}}.$$

Since u blows up at time $T_+ = 1$, this is precisely the proved blowup rate.

Furthermore, if we take instead the initial condition $u(0) = -1$, then this has the explicit solution $|u|^{1-p} = (-1 - t)$. This blows up at $T_- = -1$ and has proved blowup rate.

1.12. Let $g(t) = \log(3 + |u(t)|^2)$. Differentiating this yields the inequality

$$\begin{aligned}
\partial_t g(t) &= (3 + |u(t)|^2)^{-1}\langle F(u(t)), u(t)\rangle \\
&\lesssim (3 + |u(t)|^2)^{-1}|u(t)|(1 + |u(t)|)\log(2 + |u(t)|) \\
&\lesssim \frac{|u(t)| + |u(t)|^2}{3 + |u(t)|^2} \cdot \log(3 + |u(t)|^2) \\
&\lesssim g(t).
\end{aligned}$$

The differential Gronwall inequality implies that $g(t) \leq g(t_0)e^{C(t-t_0)}$ for some $C > 0$. Taking the exponential of both sides of the above inequality

$$3 + |u(t)|^2 \leq (3 + |u(t_0)|^2)^{e^{C(t-t_0)}}.$$

[7] Both sides of the equality represent solving (1.7) forward for time $t - t_0$ with the same initial data.

[8] The right-hand side represents solving (1.7) forward for time t, while the middle expression represents solving (1.7) forward for time t, and then solving forward again with this new data by time $t' - t$.

In particular, this implies the growth bound

$$|u(t)| \lesssim \exp\{B \exp[C(t - t_0)]\},$$

where B and C are some positive constants, with B depending on $u(t_0)$. As a result, by Theorem 1.17, solutions to this ODE exist for all times.

To that see this bound is sharp, consider the equation

$$\partial_t u = (1 + u) \log(1 + u), \qquad u(0) = e - 1,$$

which has explicit solution $1 + u(t) = \exp \exp t$. In particular, this solution satisfies

$$|u(t)| \gtrsim \exp \exp(t - 0).$$

1.13. First, by a direct calculation using the chain rule, we have

$$\partial_t[H(u(t))] = \langle \partial_t u(t), dH(u(t)) \rangle = \langle F(u(t)), dH(u(t)) \rangle = G(u(t))H(u(t)).$$

As a result, considering the nonnegative function $f = H \circ u$, we have

$$\partial_t f(t) = (G \circ u)(t) f(t).$$

Since $G \circ u$ is continuous, and since $f(t)$ vanishes at some $t_0 \in I$, then the differential Gronwall inequality, applied at base point t_0, shows that $f = H \circ u$ vanishes everywhere.

Geometrically, if we interpret F as being a vector field describing the evolution of u, then the statement has the following interpretation: if the solution curve $u(t)$ begins on the level set $H = 0$, and if $\langle F, dH \rangle$ vanishes to first order in H on the level set $H = 0$, with the ratio G being continuous, [9] then $u(t)$ remains everywhere on the level set $H = 0$.

1.21. We perform a standard bootstrap argument. Define the set

$$\mathcal{A} = \{t \in I | u(t) \leq 2A\}.$$

Since $t_0 \in \mathcal{A}$, then $\mathcal{A}$ is nonempty. Moreover, since u is continuous, then $\mathcal{A}$ is closed.

Next, suppose $t \in \mathcal{A}$, let M be the maximum value of F on the closed unit ball of radius $2A$, and let $\varepsilon = A/2M$. By our assumptions, we have the estimate

$$u(t) \leq A + \varepsilon F(u(t)) \leq A + \varepsilon M \leq \frac{3}{2} A.$$

Since u is continuous, then $\mathcal{A}$ is open. Since I is connected, and since $\mathcal{A}$ is nonempty, closed, and open, then $\mathcal{A} = I$, and hence $u(t) \leq 2A$ for all $t \in I$.

For counterexamples, suppose first that ε is not small. Let $A = 1$, let $F(v) = v$, and take, for instance, $\varepsilon = 1$. Then, the assumed inequality is $u(t) \leq 1 + u(t)$, which trivially holds. Thus, u can be any positive continuous function on I, and we have no uniform bound for u.

Next, we consider the case in which u is not continuous. Let $A = 1$, and let $F(v) = v^2$, so the assumed inequality is $u(t) \leq 1 + \varepsilon[u(t)]^2$. Note that $v \leq 1 + \epsilon v^2$ holds for $v \leq 1$ and for sufficiently large v with respect to ε, say $v \geq C_\varepsilon > 2$. Thus, we can construct the discontinuous function u by fixing $t_0 < t_1 \in I$ and defining

$$u(t) = \begin{cases} 1 & t < t_1 \\ C_\epsilon & t \geq t_1. \end{cases}$$

This function clearly satisfies $u(t_0) \leq 2A = 2$ and the assumed inequality $u(t) \leq 1 + \varepsilon[u(t)]^2$, but it is also not uniformly bounded by $2 = 2A$.

[9] In particular, F is tangent to the level set $H = 0$, since dH as a vector field is normal to the level sets of H.

1.22. From Young's inequality, we have the bound

$$Bu(t)^\theta = [2^\theta B][2^{-\theta} u(t)^\theta] \leq (1-\theta) 2^{\frac{\theta}{1-\theta}} B^{\frac{1}{1-\theta}} + \theta 2^{-1} u(t).$$

Plugging this into our assumed inequality for u, then we have

$$u(t) \leq 2[1 - \theta 2^{-1}] u(t) \leq 2A + 2(1-\theta) 2^{\frac{\theta}{1-\theta}} B^{\frac{1}{1-\theta}} + 2\varepsilon F(u(t)).$$

The desired result now follows from the above and from Exercise (1.21).

1.23. Consider an open ball $U = B(u_0, \delta)$ about u_0 such that F is continuous on $\bar{U}$. Then, there is a sequence $\{F_m : \bar{U} \to \mathbb{R}\}$ of Lipschitz continuous functions such that $F_m \to F$ uniformly. [10] Since $F_m \to F$ uniformly, the F_m's are uniformly bounded; in particular, there is some $M > 0$ such that if $u \in \bar{U}$, then $|F_m(u)| \leq M$ for all m.

We can now solve using the usual Picard theory for maximal solutions

$$u_m : (T_{-,m}, T_{+,m}) \to \mathcal{D}, \qquad \partial_t u_m(t) = F_m(u_m(t)), \qquad u_m(t_0) = u_0.$$

The next goal is to show uniform control for the $T_{-,m}$'s, the $T_{+,m}$'s, and the u_m's.

Fix now a single m, and define the constant

$$\varepsilon_m = \min((2M)^{-1}\delta, T_{+,m} - t_0, t_0 - T_{-,M}).$$

For our bootstrap argument, we define

$$\mathfrak{A}_m = \{d \in [0, \varepsilon_m) \mid |u_m(t_0 + s) - u_0| \leq \delta \text{ for all } s \in [-d, d]\}.$$

By definition, $0 \in \mathfrak{A}_m$, and $\mathfrak{A}_m$ is closed. Furthermore if $d \in \mathfrak{A}_m$, then

$$|u_m(t_0 \pm d) - u_0| \leq \int_{t_0}^{t_0 \pm d} |F_m(u_m(s))| ds \leq \varepsilon_m M \leq \frac{1}{2}\delta,$$

and by continuity, it follows that a neighborhood of d in $[0, \varepsilon_m)$ is contained in $\mathfrak{A}_m$. Therefore, $\mathfrak{A}_m$ is open, so by connectedness, then $\mathfrak{A}_m = I$. Since the above holds for any arbitrary m, then we have shown that $u_m(t) \in \bar{U}$ whenever $|t - t_0| < \varepsilon_m$.

Combining the above argument with Theorem 1.17, we see that

$$\varepsilon_m = (2M)^{-1}\delta = \varepsilon, \qquad [t_0 - \varepsilon, t_0 + \varepsilon] \subseteq (T_{-,m}, T_{+,m})$$

for every m. [11] This establishes the desired uniform bounds on the $T_{-,m}$'s and $T_{+,m}$'s. The preceding bootstrap argument also yields uniform bounds for all the u_m's on $[t_0 - \varepsilon, t_0 + \varepsilon]$. Furthermore, we have the uniform bounds

$$|\partial_t u_m(t)| \leq |F_m(u_m(t))| \leq M, \qquad |t - t_0| \leq \varepsilon,$$

so the u_m's are uniformly Lipschitz and hence equicontinuous on this interval. By the Arzela-Ascoli theorem and the above bootstrap bound, restricting to a subsequence, then the u_m's converge uniformly to a continuous function

$$u : [t_0 - \varepsilon, t_0 + \varepsilon] \to \mathcal{D}, \qquad \|u - u_0\|_\infty \leq \frac{\delta}{2}.$$

Since $u_m(t) \to u(t)$ and $u_m(t_0) \to u(t_0)$, then we have

$$u(t) = u(t_0) + \int_{t_0}^t F(u(s)) ds + \lim_m \int_{t_0}^t [F_m(u_m(s)) - F(u(s))] = u(t_0) + \int_{t_0}^t F(u(s)) ds.$$

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[10] For example, this can be easily proved using the Stone-Weierstrass theorem.

[11] If, say, $T_{+,m} - t_0 \leq (2M)^{-1}\delta$, then the bootstrap bound implies that $u(t)$ is uniformly bounded for all $t \in [t_0, T_{+,m})$, which by Theorem 1.17 contradicts the maximality of $T_{+,m}$.

In particular, the limit of the integral vanishes by the dominated convergence theorem, since we have the crude bound $|F_m(u_m(s)) - F(u(s))| \leq 2M$. Finally, by Lemma 1.3, then u is a classical solution to the original ODE.

1.27. *Correction:* We also require the following orthogonality condition for J: [12]

$$\langle Ju, Jv\rangle = \langle u, v\rangle, \qquad u, v \in \mathcal{D}.$$

Without this condition, ω can fail to be antisymmetric. For example, if

$$\mathcal{D} = \mathbb{R}^2, \qquad e_1 = (1, 0), \qquad e_2 = (0, 1),$$

then we can define $Je_1 = 2e_2$ and $Je_2 = -\frac{1}{2}e_1$, so that

$$\omega(e_1, e_2) = -\frac{1}{2}\langle e_1, e_1\rangle = -\frac{1}{2}, \qquad \omega(e_2, e_1) = 2\langle e_2, e_2\rangle = 2.$$

With the above condition, first note that ω is indeed antisymmetric, since

$$\omega(v, u) = \langle Ju, v\rangle = \langle J^2u, Jv\rangle = -\langle u, Jv\rangle = -\omega(u, v).$$

To show bilinearity, we need only show linearity in the first variable: [13]

$$\omega(au_1 + bu_2, v) = \langle au_1 + bu_2, Jv\rangle = a\langle u_1, Jv\rangle + b\langle u_2, Jv\rangle = a\omega(u_1, v) + b\omega(u_2, v).$$

For nondegeneracy, given $u \in \mathcal{D} \setminus \{0\}$, then $\langle u, u\rangle > 0$, so that $\omega(u, Ju) = -\langle u, u\rangle \neq 0$. As a result, ω is a symplectic form.

Finally, to see that $\nabla_\omega H = J\nabla H$, [14] we note that for any $v \in \mathcal{D}$,

$$\omega(J\nabla H, v) = \langle J\nabla H, Jv\rangle = \langle \nabla H, v\rangle.$$

1.28. We induct on the dimension n of $\mathcal{D}$. First of all, the desired statement is trivial for dimension $n = 0$. Fix now $n > 0$, and suppose the conclusion is true for any dimension strictly less than n. Then, we need only prove the same conclusion for dimension n.

Let $u \in \mathcal{D} \setminus \{0\}$. Since ω is nondegenerate, there is some $v \in \mathcal{D}$ such that $\omega(u, v) = 1$. [15] In addition, the restriction ω_0 of ω to the subspace $\mathcal{D}_0 = \text{span}\{u, v\}$ is $du \wedge dv$, i.e.,

$$\omega(a_1u + b_1v, a_2u + b_2v) = a_1b_2 - a_2b_1.$$

Consider next the symplectic complement

$$\mathcal{D}' = \{w \in \mathcal{D} | \omega(u, w) = \omega(v, w) = 0\}.$$

Since ω is nondegenerate, then the linear functionals $\omega_u = \omega(u, \cdot)$ and $\omega_v = \omega(v, \cdot)$ map onto $\mathbb{R}$, so that their nullspaces satisfy

$$\dim \mathcal{N}(\omega_u) = \dim \mathcal{N}(\omega_v) = n - 1.$$

Furthermore, since ω_{u-v} is nontrivial as well, then $\mathcal{N}(\omega_u)$ and $\mathcal{N}(\omega_v)$ cannot completely coincide, and it follows that

$$\dim \mathcal{D}' = \dim[\mathcal{N}(\omega_u) \cap \mathcal{N}(\omega_v)] = n - 2.$$

By the induction hypothesis, then the restriction ω' to $\mathcal{D}'$ has the desired standard decomposition in some coordinates p_j and q_j, where $1 \leq j \leq n/2 - 1$. In other words,

$$\omega' = \sum_{1 \leq i \leq \frac{n-2}{2}} (dq_j \wedge dp_j).$$

[12] This may already have been covered by the condition that J is an endomorphism of $\mathcal{D}$.

[13] Linearity in the second variable follows immediately from the antisymmetry of ω.

[14] Although the book asserts $\nabla_\omega H = -J\nabla H$, the minus sign should not be present here.

[15] In particular, $n \geq 2$.

In particular, both $\mathcal{D}'$ and $\mathcal{D}$ have even dimension. Since $\mathcal{D}'$ and $\mathcal{D}_0$ are by definition ω-orthogonal, then ω in fact also has this form:

$$\omega = \sum_{1 \leq i \leq \frac{n-2}{2}} (dq_j \wedge dp_j) + du \wedge dv.$$

1.30. Suppose u satisfies a Hamiltonian equation, with Hamiltonian H. We make the change of variables $v(t) = u(-t)$. By the chain rule,

$$\partial_t v(t) = -\nabla_\omega H(u(-t)) = -\nabla_\omega H(v(t)),$$

and hence v also satisfies a Hamiltonian equation, with Hamiltonian $-H$.

1.31. We can define a natural product $\langle \cdot, \cdot \rangle_{\mathcal{D}\times\mathcal{D}'}$ and symplectic form $\omega \oplus \omega'$ on $\mathcal{D} \times \mathcal{D}'$ by

$$\langle (u, u'), (v, v') \rangle_{\mathcal{D}\times\mathcal{D}'} = \langle u, v \rangle_{\mathcal{D}} + \langle u', v' \rangle_{\mathcal{D}'},$$
$$(\omega \oplus \omega')((u, u'), (v, v')) = \omega(u, v) + \omega'(u', v').$$

Define the Hamiltonion $H \oplus H' \in C^2(\mathcal{D} \times \mathcal{D}' \to \mathbb{R})$ by

$$(H \oplus H')(u, u') = H(u) + H'(u').$$

A standard calculation yields that its differential is

$$d(H \oplus H')(u, u') = (dH(u), dH'(u')) \in \mathcal{D} \times \mathcal{D}',$$

so that

$$\begin{aligned} \langle d(H \oplus H')(u, u'), (v, v') \rangle_{\mathcal{D}\times\mathcal{D}'} &= \langle dH(u), v \rangle_{\mathcal{D}} + \langle dH'(u'), v' \rangle_{\mathcal{D}'} \\ &= \omega(\nabla_\omega H(u), v) + \omega'(\nabla_{\omega'} H'(u'), v') \\ &= (\omega \oplus \omega')((\nabla_\omega H(u), \nabla_{\omega'} H'(u')), (v, v')). \end{aligned}$$

As a result,

$$\nabla_{\omega\oplus\omega'}(H \oplus H')(u, u') = (\nabla_\omega H(u), \nabla_{\omega'} H'(u')),$$

and hence u and u', as given in the problem, satisfy

$$\partial_t(u(t), u'(t)) = (\nabla_\omega H(u(t)), \nabla_{\omega'} H'(u'(t))) = \nabla_{\omega\oplus\omega'}(H \oplus H')(u(t), u'(t)).$$

1.32. Let $\dim \mathcal{D} = 2n$, and let p_i, q_i, where $1 \leq i \leq n$, denote the standard coordinates for the symplectic space $(\mathcal{D}, \omega)$; see Example (1.27) and Exercise (1.28).

First, suppose $u \in C^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathcal{D})$, such that $u(\cdot, x, y)$ is a solution curve for the Hamiltonian equation for H for every $x, y \in \mathbb{R}$, i.e., that

$$\partial_t u(t, x, y) = \nabla_\omega H(u(t, x, y)), \qquad t, x, y \in \mathbb{R}.$$

By a direct computation, we have

$$\begin{aligned} \partial_t[\omega(\partial_x u(t, x, y), \partial_y u(t, x, y))] &= \omega(\partial_x \partial_t u(t, x, y), \partial_y u(t, x, y)) + \omega(\partial_x u(t, x, y), \partial_y \partial_t u(t, x, y)) \\ &= \omega(\partial_x \nabla_\omega H(u(t, x, y)), \partial_y u(t, x, y)) \\ &\quad + \omega(\partial_x u(t, x, y), \partial_y \nabla_\omega H(u(t, x, y))), \end{aligned}$$

where in the first equality, one can justify the Leibniz rule for ∂_t by expanding ω in terms of the q_i's and p_i's. By the bilinearity properties of ω, along with the definition of ∇_ω, then

$$\begin{aligned} \partial_t[\omega(\partial_x u(t, x, y), \partial_y u(t, x, y))] &= \langle \partial_x[dH(u(t, x, y))], \partial_y u(t, x, y) \rangle \\ &\quad - \langle \partial_x u(t, x, y), \partial_y[dH(u(t, x, y))] \rangle \\ &= \langle \nabla^2 H(u(t, x, y))[\partial_x u(t, x, y)], \partial_y u(t, x, y) \rangle \\ &\quad - \langle \partial_x u(t, x, y), \nabla^2 H(u(t, x, y))[\partial_y u(t, x, y)] \rangle, \end{aligned}$$

where in the last step, we simply applied the chain rule. Treating the Hessian $\nabla^2 H$ of H at $u(t, x, y)$ as a bilinear map, then the above becomes

$$\partial_t[\omega(\partial_x u(t, x, y), \partial_y u(t, x, y))] = \nabla^2 H(u(t, x, y))[\partial_x u(t, x, y), \partial_y u(t, x, y)] - \nabla^2 H(u(t, x, y))[\partial_y u(t, x, y), \partial_x u(t, x, y)],$$

which of course vanishes. As a result, $\omega(\partial_x u(t, x, y), \partial_y u(t, x, y))$ is conserved in time.

Furthermore, in the quadratic growth case, in which $\nabla^2 H$ is bounded, then from the discussions after Example (1.28), we know that the H-Hamiltonian equation always has global solutions. Thus, the solution maps $S(t)$ are always well-defined for all $t \in \mathbb{R}$.

Elaboration: To show that the solution maps are symplectomorphisms, we must first provide some additional background detailing how this symplectic form ω is preserved by the Hamiltonian evolution. Consider the vector space $\mathcal{D}$ as a $2n$-dimensional real manifold; recall that each tangent space $T_x\mathcal{D}$, where $x \in \mathcal{D}$, can be identified with $\mathcal{D}$. [16] Then, we can impose a symplectic form $\bar{\omega}$ on the *manifold* $\mathcal{D}$ such that at each $T_x\mathcal{D}$, the bilinear form $\bar{\omega}|_x$ is identified with ω according to the above identification of $T_x\mathcal{D}$ and $\mathcal{D}$.

For any $t \in \mathbb{R}$, the pullback $S(t)^*\bar{\omega}$ of $\bar{\omega}$ through the solution map $S(t)$ defines a differential form on (the manifold) $\mathcal{D}$. Our goal is to show that $S(t)^*\bar{\omega} = \bar{\omega}$. Let X, Y be arbitrary vector fields on $\mathcal{D}$, with coordinate decompositions $X = X^\alpha \partial_\alpha$ and $Y = Y^\beta \partial_\beta$ with respect to the standard coordinates. Define also the map

$$u : \mathbb{R} \times \mathcal{D} \to \mathcal{D}, \qquad u(t, x) = S(t)x.$$

Letting $dS(t)$ denote the standard differential map of $S(t)$, then we can compute

$$S(t)^*\bar{\omega}(X, Y) = \bar{\omega}([dS(t)]X, [dS(t)]Y) = X^\alpha Y^\beta \cdot \bar{\omega}(\partial_\alpha(y^\gamma \circ S(t))\partial_\gamma, \partial_\beta(y^\delta \circ S(t))\partial_\delta),$$

where the y^γ's represent the standard coordinate functions p_i and q_j. Recalling the pointwise identification between $\bar{\omega}$ and ω, then we have

$$S(t)^*\bar{\omega}(X, Y)|_x = X^\alpha Y^\beta \omega(\partial_\alpha(S(t)x), \partial_\beta(S(t)y)) = X^\alpha Y^\beta \omega(\partial_\alpha u(t, x), \partial_\beta u(t, x)).$$

From our conservation property for ω and u, proved in the beginning of this exercise, then

$$S(t)^*\bar{\omega}(X, Y)|_x = X^\alpha Y^\beta \omega(\partial_\alpha u(0, x), \partial_\beta u(0, x)) = S(0)^*\bar{\omega}(X, Y) = \bar{\omega}(X, Y).$$

The second to last step follows from the same computations and identifications as before, and the last step is simply because $S(0)$ is the identity map. As a result, we have shown that $S(t)$ is indeed a symplectomorphism, as desired.

1.33. Consider the Liouville measure on $\mathcal{D}$ defined by the top form

$$m \approx \omega^n = \omega \wedge \cdots \wedge \omega \quad (n \text{ times}),$$

that is, the measure defined

$$m(\Omega) = \int_\Omega \omega^n.$$

Our goal is to show that $m(S(t)(\Omega))$ is the same as $m(\Omega)$ for any Lebesgue measurable $\Omega \subseteq \mathcal{D}$. By a change of variables, we have

$$m(S(t)(\Omega)) = \int_{S(t)(\Omega)} \omega^n = \int_\Omega S(t)^*\omega^n.$$

By Exercise (1.32) and standard properties of pullback forms, then

$$S(t)^*\omega^n = [S(t)^*\omega]^n = \omega^n,$$

[16] The component vectors $q_i, p_j \in \mathcal{D}$ are identified with the tangent vectors $\partial_{q_i}|_x, \partial_{p_j}|_x \in T_x\mathcal{D}$, respectively.

and it follows that

$$m(S(t)(\Omega)) = \int_{\Omega} \omega^n = m(\Omega).$$

Finally, since H is conserved by the solution map, then for any $t \in \mathbb{R}$,

$$S(t)^*[e^{-\beta H}\omega^n] = e^{-\beta[H \circ S(t)]} S(t)^* \omega^n = e^{-\beta[H\circ S(0)]}\omega^n = e^{-\beta H}\omega^n.$$

As a result, letting $d\mu_\beta = e^{-\beta H} dm$ denote the Gibbs measure, we have

$$d\mu_\beta(S(t)(\Omega)) = \int_{S(t)(\Omega)} e^{-\beta H}\omega^n = \int_{\Omega} S(t)^*[e^{-\beta H}\omega^n] = \int_{\Omega} e^{-\beta H}\omega^n = d\mu_\beta(\Omega).$$

1.35. For the Jacobi identity, we break down into "standard" coordinates

$$(q_1, \dots, q_n; p_1, \dots, p_n), \qquad n = \dim \mathcal{D},$$

described in Exercise 1.28. From computations in Example 1.27, we see that

$$\begin{aligned}
\{H_1, \{H_2, H_3\}\} &= \sum_{j=1}^{n} (\partial_{q_j} H_1 \partial_{p_j} \{H_2, H_3\} - \partial_{p_j} H_1 \partial_{q_j} \{H_2, H_3\}) \\
&= \sum_{j=1}^{n} \sum_{l=1}^{n} (\partial_{q_j} H_1 \partial_{p_j} - \partial_{p_j} H_1 \partial_{q_j})(\partial_{q_l} H_2 \partial_{p_l} H_3 - \partial_{p_l} H_2 \partial_{q_l} H_3) \\
&= \sum_{j,l=1}^{n} (\partial_{q_j} H_1 \partial_{p_j q_l} H_2 \partial_{p_l} H_3 + \partial_{q_j} H_1 \partial_{q_l} H_2 \partial_{p_j p_l} H_3) \\
&\quad - \sum_{j,l=1}^{n} (\partial_{q_j} H_1 \partial_{p_j p_l} H_2 \partial_{q_l} H_3 + \partial_{q_j} H_1 \partial_{p_l} H_2 \partial_{p_j q_l} H_3) \\
&\quad - \sum_{j,l=1}^{n} (\partial_{p_j} H_1 \partial_{q_j q_l} H_2 \partial_{p_l} H_3 + \partial_{p_j} H_1 \partial_{q_l} H_2 \partial_{q_j p_l} H_3) \\
&\quad + \sum_{j,l=1}^{n} (\partial_{p_j} H_1 \partial_{q_j p_l} H_2 \partial_{q_l} H_3 + \partial_{p_j} H_1 \partial_{p_l} H_2 \partial_{q_j q_l} H_3).
\end{aligned}$$

The brackets $\{H_2, \{H_3, H_1\}\}$ and $\{H_3, \{H_1, H_2\}\}$ have similar expansions, but with the H_i's permuted. Summing these expansions, we can see that all the individual terms cancel.

Next, for the Leibniz rule, we first note that

$$\begin{aligned}
\langle d(H_1 H_2)(u), v\rangle &= H_1(u)\langle dH_2(u), v\rangle + H_2\langle dH_1(u), v\rangle \\
&= \omega(H_1(u)\nabla_\omega H_2(u), v) + \omega(H_2(u)\nabla_\omega H_1(u), v).
\end{aligned}$$

In other words, the symplectic gradient satisfies the product rule:

$$\nabla_\omega(H_1 H_2) = H_1 \nabla_\omega H_2 + H_2 \nabla_\omega H_1.$$

As a result, we can compute as desired

$$\begin{aligned}
\{H_1, H_2 H_3\} &= \omega(\nabla_\omega H_1, \nabla_\omega H_2) H_3 + \omega(\nabla_\omega H_1, \nabla_\omega H_3) H_2 \\
&= \{H_1, H_2\} H_3 + H_2 \{H_1, H_3\}.
\end{aligned}$$

1.36. We can compute this using the Jacobi identity from Exercise 1.35:

$$[D_{H_1}, D_{H_2}]E = \{H_1, \{H_2, E\}\} - \{H_2, \{H_1, E\}\} = -\{E, \{H_1, H_2\}\} = D_{\{H_1, H_2\}} E.$$

1.37. First of all, if E and H do not Poisson commute, then by the identity (1.33), there is a solution curve u to (1.28) for which $(E \circ u)' = \{E, H\} \circ u$ has constant nonzero sign on some small interval $[t_0, t_1]$. As a result, $E(u(t_1)) - E(u(t_0))$ is nonzero, but the integral

$$\int_{t_0}^{t_1} G(u(t))[\partial_t u(t) - \nabla_\omega H(u(t))] dt$$

vanishes for any $G \in C^0_{loc}(\mathcal{D} \to \mathcal{D}^*)$, so that E cannot be an integral of motion.

On the other hand, if E and H do Poisson commute, then on any interval $[t_0, t_1]$, and for any C^1 curve $u : [t_0, t_1] \to \mathcal{D}$, we have the identity

$$\begin{aligned} E(u(t_1)) - E(u(t_0)) &= \int_{t_0}^{t_1} (E \circ u)'(t) dt \\ &= \int_{t_0}^{t_1} \langle dE(u(t)), \partial_t u(t) \rangle dt \\ &= \int_{t_0}^{t_1} [\langle dE(u(t)), \partial_t u(t) - \nabla_\omega H(u(t)) \rangle + \langle dE(u(t)), \nabla_\omega H(u(t)) \rangle] dt \\ &= \int_{t_0}^{t_1} \langle dE(u(t)), \partial_t u(t) - \nabla_\omega H(u(t)) \rangle dt + \int_{t_0}^{t_1} \{E, H\}(u(t)) dt. \end{aligned}$$

By our assumption, the last term on the right-hand side vanishes, and it follows that E is indeed an integral of motion of (1.28).

1.42. We consider the symplectic space $(\bar{\mathcal{D}}, \bar{\omega})$, where [17]

$$\bar{\mathcal{D}} = \mathbb{R} \times \mathbb{R} \times \mathcal{D}, \qquad \bar{\omega}((q_1, p_1, u_1), (q_2, p_2, u_2)) = q_1 p_2 - p_1 q_2 + \omega(u_1, u_2).$$

A quick computation shows that the $\bar{\omega}$-symplectic gradient is given by

$$\nabla_{\bar{\omega}} f = (\partial_p f, -\partial_q f, \nabla_{u,\omega} f), \qquad f \in C^2(\bar{\mathcal{D}} \to \mathbb{R}).$$

In the above, p, q, and u refer to the first, second, and third arguments of f, respectively, while $\nabla_{u,\omega} f$ refers to the ω-symplectic gradient of f with respect to the u-variable.

Consider the time-dependent Hamiltonian and the associated Hamiltonian equation

$$H \in C^1(\mathbb{R} \times \mathcal{D} \to \mathbb{R}), \qquad \partial_t u(t) = \nabla_{u,\omega} H(t, u(t)),$$

where in the above, "$\nabla_{u,\omega} H$" refers to the ω-symplectic gradient with respect to the second argument of H. Consider the following *time-independent* Hamiltonian on $\bar{\mathcal{D}}$:

$$\bar{H} \in C^1(\bar{\mathcal{D}} \to \mathbb{R}), \qquad \bar{H}(q, p, u) = H(q, u) + p.$$

A quick computation shows that

$$\nabla_{\bar{\omega}} \bar{H}(q, p, u) = (1, -\partial_q H(q, u), \nabla_{u,\omega} H(q, u)).$$

We can now consider the (time-independent) Hamiltonian equation

$$\partial_t \begin{bmatrix} q(t) \\ p(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} 1 \\ \partial_q H(q(t), u(t)) \\ \nabla_{u,\omega} H(q(t), u(t)) \end{bmatrix}, \qquad (q(t_0), p(t_0), u(t_0)) = (t_0, 0, u_0) \in \bar{\mathcal{D}}.$$

where $(q, p, u) \in C^1(\mathbb{R} \to \bar{\mathcal{D}})$. We can immediately solve the above for q, which yields $q(t) = t$. As a result, then u solves the time-dependent Hamiltonian equation

$$\partial_t u(t) = \nabla_{u,\omega} H(t, u(t)).$$

[17]In other words, we define $\bar{\omega}$ by combining ω with the symplectic form in Example (1.27).

Thus, the above time-dependent Hamiltonian setting can be reformulated as an equivalent time-independent Hamiltonian setting. Furthermore, note that p satisfies

$$p(t) = -\int_{t_0}^{t} \partial_q H(q(s), u(s)) ds = -\int_{t_0}^{t} \partial_q H(s, u(s)) ds.$$

For such a time-dependent Hamiltonian H, the associated Hamiltonian equation needs not preserve H. For example, if $H(t, u) = t$, then we have the equation

$$\partial_t u(t) = \nabla_{u,\omega} t \equiv 0, \qquad u(t_0) = u_0,$$

which has trivial solution $u(t) \equiv u_0$. However, H fails to be constant in time, since

$$H(t, u(t)) = H(t, u_0) = t.$$

However, by the time-independent Hamiltonian theory, then $\bar{H}$ is preserved by solution curves of the $\bar{H}$-Hamiltonian equation. Thus, a substitute quantity for the time-dependent H-Hamiltonian equation that is preserved by its solution curves is

$$\bar{H}(q(t), p(t), u(t)) = H(t, u(t)) + p(t) = H(t, u(t)) - \int_{t_0}^{t} \partial_q H(s, u(s)) ds.$$

1.44. With H defined in terms of L as above, we can first compute the partial derivatives of H. First of all, $\dot{q}$ is by definition a function of both p and q, so that

$$\begin{aligned}
\partial_{q_i} H(q, p) &= \sum_{j=1}^{n} (\partial_{q_i} \dot{q}_j \cdot p_j) - \partial_{q_i} L(q, \dot{q}) - \sum_{j=1}^{n} \partial_{\dot{q}_j} L(q, \dot{q}) \cdot \partial_{q_i} \dot{q}_j \\
&= \sum_{j=1}^{n} \partial_{q_i} \dot{q}_j \cdot [p_j - \partial_{\dot{q}_j} L(q, \dot{q})] - \partial_{q_i} L(q, \dot{q}) \\
&= -\partial_{q_i} L(q, \dot{q}).
\end{aligned}$$

In the last step, we applied (1.37). By a similar computation, we also have

$$\partial_{p_i} H(q, p) = \dot{q}_i + \sum_{j=1}^{n} \partial_{p_i} \dot{q}_j \cdot p_j - \sum_{j=1}^{n} \partial_{p_i} \dot{q}_j \cdot \partial_{\dot{q}_j} L(q, \dot{q}) = \dot{q}_i.$$

Thus, the Hamiltonian equation is

$$\partial_t q_i(t) = \partial_{p_i} H(q(t), p(t)) = \dot{q}_i(t), \qquad \partial_t p_i(t) = -\partial_{q_i} H(q(t), p(t)) = \partial_{q_i} L(q(t), \dot{q}(t)).$$

Now, fix a curve $q \in C^\infty(I \to \mathbb{R}^n)$ as in the problem statement. In addition, we define the "momentum" curve $p \in C^\infty(I \to \mathbb{R}^n)$ as in (1.37):

$$p_j(t) = \partial_{\dot{q}_j} L(q(t), \partial_t q(t)).$$

Note that in this setup, the first Hamiltonian is trivially satisfied.

Consider now a variation $q + \varepsilon v$, with ε sufficiently small, and with $v \in C^\infty(I \to \mathbb{R}^n)$ vanishing at the endpoints of I. Then, we can compute

$$\begin{aligned}
\frac{d}{d\varepsilon} S(q(t) + \varepsilon v(t))|_{\varepsilon=0} &= \frac{d}{d\varepsilon} \int_I L(q(t) + \varepsilon v(t), \partial_t q(t) + \varepsilon \partial_t v(t)) dt|_{\varepsilon=0} \\
&= \sum_{i=1}^{n} \int_I [\partial_{q_i} L(q(t), \partial_t q(t)) \cdot v(t) + \partial_{\dot{q}_i} L(q(t), \partial_t q(t)) \cdot \partial_t v(t)] dt \\
&= \sum_{i=1}^{n} \int_I [\partial_{q_i} L(q(t), \partial_t q(t)) \cdot v(t) + p_i(t) \cdot \partial_t v(t)] dt
\end{aligned}$$

$$= \int_I \sum_{i=1}^n [-\partial_t p_i(t) + \partial_{q_i} L(q(t), \partial_t q(t))] \cdot v(t) dt,$$

where in the last step, we integrated by parts to treat the derivative $\partial_t v(t)$. Since q is a critical point of the Lagrangian if and only if the left-hand side vanishes for all such curves v, then by the above computation, this happens if and only if

$$\partial_t p_i(t) = \partial_{q_i} L(q(t), \partial_t q(t)), \qquad 1 \leq i \leq n.$$

This is precisely the remaining half of the Hamiltonian equation.

1.45. Consider a variation $(q + \varepsilon v, p + \varepsilon w)$ of (q, p), where $\varepsilon > 0$ is small, and where $v, w \in C^\infty(I \to \mathbb{R}^n)$ vanish on the endpoints of I. Defining $\mathcal{S}$ to be the variation

$$\mathcal{S} = \frac{d}{d\varepsilon} S(q(t) + \varepsilon v(t), p(t) + \varepsilon w(t))|_{\varepsilon=0},$$

then we can compute

$$\begin{aligned} \mathcal{S} &= \frac{d}{d\varepsilon} \int_I [\partial_t q(t) + \varepsilon \partial_t v(t)][p(t) + \varepsilon w(t)] - H(q(t) + \varepsilon v(t), p(t) + \varepsilon w(t)) dt|_{\varepsilon=0} \\ &= \int_I [\partial_t q(t) w(t) + p(t) \partial_t v(t) - v(t) \cdot \nabla_q H(q(t), p(t)) - w(t) \cdot \nabla_p H(q(t), p(t))] dt \\ &= \int_I (\partial_t q(t) - \nabla_p H(q(t), p(t)), -\partial_t p(t) - \nabla_q H(q(t), p(t))) \cdot (w(t), v(t)) dt, \end{aligned}$$

where in the last step, we handled the derivative $\partial_t v(t)$ by integrating by parts. Therefore, (q, p) is a critical point of this Lagrangian if and only if the above vanishes for all such v and w. This happens if and only if the Hamiltonian equations for (q, p):

$$\partial_t q_i(t) = \partial_{p_i} H(q(t), p(t)), \qquad \partial_t p_i(t) = -\partial_{q_i} H(q(t), p(t)), \qquad 1 \leq i \leq n.$$

This is essentially the inverse to Exercise 1.44. Both exercises assert that one has a critical point with respect to a Lagrangian if and only if the associated Hamiltonian equations hold. The difference is that Exercise 1.44 states this with respect to L, q, and $\dot{q} = \partial_t q$, while Exercise 1.45 states this in terms of H, q, and p. In particular, this demonstrates the invertible relationships between p and $\dot{q}$ and between H and L:

$$\begin{aligned} p_j &= \partial_{\dot{q}_j}(q, \dot{q}), & \dot{q}_j &= \partial_{p_j} H(q, p), \\ H(q, p) &= \dot{q} \cdot p - L(q, \dot{q}), & L(q, \dot{q}) &= \dot{q} \cdot p - H(q, p). \end{aligned}$$

1.46. Suppose x is a maximal solution for (1.39), defined on the interval $I = (T_-, T_+)$, which contains $t_0 = 0$. Since $V \geq 0$, then for any $t \in I$, we have

$$|\partial_t x(t)| \leq \sqrt{2E(t)} = \sqrt{2E(0)} < \infty.$$

Here, we recalled that the energy $E(t)$, defined in (1.40), is conserved. [18] Moreover,

$$|x(t)| \leq |x(0)| + |t| \sup_{s \in I} |\partial_t x(s)| \leq |x(0)| + |t| \sqrt{2E(0)} < \infty, \qquad t \in I.$$

Thus, by Theorem 1.17, we have $T_\pm = \pm\infty$, i.e., x is a C^2 global solution.

Next, suppose $x(t_0) = 0$ for some $t_0 \in \mathbb{R}$, and suppose $x(t_1) = 0$ for some $t_1 \neq t_0$. Since $|x|^2$ is convex (see Example 1.31), then we obtain for any $0 \leq \alpha \leq 1$ that

$$|x(\alpha t_0 + (1 - \alpha) t_1)|^2 \leq \alpha |x(t_0)|^2 + (1 - \alpha) |x(t_1)|^2 = 0.$$

[18] See the proof of Proposition 1.24.

Thus, x vanishes for all times between t_0 and t_1, and by uniqueness, $x \equiv 0$ everywhere. As a result, if x does not vanish everywhere, then x can hit zero at at most one point t_0.

Finally, suppose $x(t_0) = 0$ for some $t_0 \in \mathbb{R}$, and let

$$P(t) = \frac{|\pi_{x(t)} \partial_t x(t)|^2}{|x(t)|} - \frac{x(t) \cdot \nabla V(x(t))}{|x(t)|} = \partial_t \left[\frac{x(t)}{|x(t)|} \cdot \partial_t x(t) \right] = \partial_t Q(t),$$

for any $t \in \mathbb{R} \setminus \{t_0\}$. [19] By the fundamental theorem of calculus, for large $R > 0$,

$$\int_{t_0+\varepsilon}^{R} P(t) dt + \int_{-R}^{t_0-\varepsilon} P(t) dt + Q(t_0 + \varepsilon) - Q(t_0 - \varepsilon) = Q(R) - Q(-R).$$

Since V is radially decreasing, then $P \geq 0$, and hence

$$\int_{t_0+\varepsilon}^{R} P(t) dt + \int_{-R}^{t_0-\varepsilon} P(t) dt + Q(t_0 + \varepsilon) - Q(t_0 - \varepsilon) \leq 2 \sup_{s \in \mathbb{R}} |Q(s)| \leq 2\sqrt{2E}.$$

As the above holds uniformly for all R, then letting $R \nearrow \infty$ yields

$$\int_{\mathbb{R} \setminus (t_0 - \varepsilon, t_0 + \varepsilon)} P(t) dt + Q(t_0 + \varepsilon) - Q(t_0 - \varepsilon) \leq 2\sqrt{2E}.$$

It remains to compute the limits of $Q(t)$ as $t \to t_0$. First, we have

$$\lim_{t \searrow t_0} \frac{x(t) \cdot \partial_t x(t)}{|x(t)|} = \lim_{t \searrow 0} \left| \frac{t - t_0}{x(t)} \right| \cdot \frac{x(t)}{t - t_0} \cdot \partial_t x(t) = |\partial_t x(t)|.$$

An analogous calculation yields

$$\lim_{t \nearrow t_0} \frac{x(t) \cdot \partial_t x(t)}{|x(t)|} = - \lim_{t \nearrow 0} \left| \frac{t - t_0}{x(t)} \right| \cdot \frac{x(t)}{t - t_0} \cdot \partial_t x(t) = -|\partial_t x(t)|.$$

As a result, letting $\varepsilon \searrow 0$ in the previous inequality, we obtain as desired

$$\int_{\mathbb{R}} P(t) dt + 2|\partial_t x(t_0)| \leq 2\sqrt{2E}.$$

1.49. Define the map

$$\varphi : B_\varepsilon \to \mathcal{S}, \qquad \varphi(v) = u_{\text{lin}} + DN(v).$$

Note that $v \in B_\varepsilon$ solves (1.50) if and only if v is a fixed point of φ. If $v \in B_\varepsilon$, then

$$\|\varphi(v)\|_{\mathcal{S}} \leq \|u_{\text{lin}}\|_{\mathcal{S}} + \|DN(v)\|_{\mathcal{S}} \leq \frac{\varepsilon}{2} + C_0 \|N(v) - N(0)\|_{\mathcal{N}} \leq \frac{\varepsilon}{2} + \frac{1}{2}\|v - 0\|_{\mathcal{S}} \leq \varepsilon.$$

hence φ maps B_ε into itself. Next, φ is a contraction mapping, since for any $u, v \in B_\varepsilon$,

$$\|\varphi(u) - \varphi(v)\|_{\mathcal{S}} = \|DN(u) - DN(v)\|_{\mathcal{N}} \leq C_0 \|N(u) - N(v)\|_{\mathcal{N}} \leq \frac{1}{2}\|u - v\|_{\mathcal{S}}.$$

Since B_ε is a closed subset of $\mathcal{S}$, it is a complete metric space. By the contraction mapping theorem, φ has a unique fixed point $u \in B_\epsilon$, which is the unique solution in B_ε to (1.50).

Lastly, let $u, v \in B_\varepsilon$ denote two solutions to (1.50), with their respective "linear parts" denoted by $u_{\text{lin}}, v_{\text{lin}} \in B_{\varepsilon/2}$. Then, by our assumptions, we have the estimate

$$\|u - v\|_{\mathcal{S}} \leq \|u_{\text{lin}} - v_{\text{lin}}\|_{\mathcal{S}} + \|DN(u) - DN(v)\|_{\mathcal{S}} \leq \|u_{\text{lin}} - v_{\text{lin}}\|_{\mathcal{S}} + \frac{1}{2}\|u - v\|_{\mathcal{S}}.$$

Rearranging the terms, we obtain

$$\frac{1}{2}\|u - v\|_{\mathcal{S}} \leq \|u_{\text{lin}} - v_{\text{lin}}\|_{\mathcal{S}}, \qquad \|u - v\|_{\mathcal{S}} \leq 2\|u_{\text{lin}} - v_{\text{lin}}\|_{\mathcal{S}}.$$

[19] See Example 1.32 for details behind this computation.

The desired Lipschitz estimate for the "solution map" $u_{\text{lin}} \mapsto u$ follows. Applying the above estimate to the special case $v = v_{\text{lin}} = 0$ (the trivial solution) yields

$$\|u\|_{\mathcal{S}} \leq 2\|u_{\text{lin}}\|_{\mathcal{S}}.$$

1.50. Since $u = u_{\text{lin}} + DN(u)$, then

$$\tilde{u} - u = DN(\tilde{u}) - DN(u) + e.$$

As a result, by (1.51) and (1.52),

$$\|\tilde{u} - u\|_{\mathcal{S}} \leq \|e\|_{\mathcal{S}} + \|DN(\tilde{u}) - DN(u)\|_{\mathcal{S}} \leq \|e\|_{\mathcal{S}} + \frac{1}{2}\|\tilde{u} - u\|_{\mathcal{S}}.$$

The desired estimate follows.

1.51. *Correction:* The correct assumption needed for ε is

$$\varepsilon^{k-1} = \frac{1}{2kC_0C_1}.$$

We begin by using the triangle inequality and expanding

$$\begin{aligned}\|N(u) - N(v)\|_{\mathcal{N}} &\leq \|N_k(u - v, u, \dots, u)\|_{\mathcal{N}} + \|N_k(v, u - v, u, \dots, u)\|_{\mathcal{N}} \\ &\quad + \dots + \|N_k(v, \dots, v, u - v)\|_{\mathcal{N}} \\ &\leq C_1\|u - v\|_{\mathcal{S}}(\|u\|_{\mathcal{S}}^{k-1} + \|v\|_{\mathcal{S}}\|u\|_{\mathcal{S}}^{k-2} + \dots + \|v\|_{\mathcal{S}}^{k-1}) \\ &\leq kC_1\varepsilon^{k-1}\|u - v\|_{\mathcal{S}}.\end{aligned}$$

Thus, if the above assumption for ε holds, then

$$\|N(u) - N(v)\|_{\mathcal{N}} \leq \frac{1}{2C_0}\|u - v\|_{\mathcal{S}}.$$

1.52. We reduce the second-order ODEs, both homogeneous and nonhomogeneous, to an equivalent first-order systems by setting $v = \partial_t u$. In other words, we consider the system

$$\partial_t u = v, \quad \partial_t v = Lu + f, \qquad u(t_0) = u_0, \quad v(t_0) = u_1.$$

If we define the linear operator

$$\tilde{L} : \mathcal{D} \times \mathcal{D} \to \mathcal{D} \times \mathcal{D}, \qquad \tilde{L}(u, v) = (v, Lu)$$

and the map

$$\tilde{f} : \mathbb{R} \to \mathcal{D} \times \mathcal{D}, \qquad \tilde{f} = (0, f),$$

then we can rewrite the above system as

$$\partial_t(u, v) = \tilde{L}(u, v) + \tilde{f}, \qquad (u, v)(t_0) = (u_0, u_1).$$

First, consider the linear case $f \equiv 0$ (i.e., $\tilde{f} \equiv 0$), from which we have

$$(u, v)(t) = e^{(t-t_0)\tilde{L}}(u_0, u_1).$$

Taking the first component of the above, then we see that there exist operators

$$U_i : \mathbb{R} \times \mathcal{D} \to \mathcal{D}, \qquad i \in \{0, 1\},$$

such that

$$u(t) = U_0(t - t_0)u_0 + U_1(t - t_0)u_1.$$

More specifically, if we expand $\exp(t\tilde{L})$ as a 2×2 matrix

$$\begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix},$$

then $U_0(t) = a_{11}(t)$ and $U_1(t) = a_{12}(t)$.

Next, for general f, then Duhamel's principle (Proposition 1.35) yields

$$(u, v)(t) = e^{(t-t_0)\tilde{L}}(u_0, u_1) + \int_{t_0}^{t} e^{(t-s)\tilde{L}}(0, f) ds.$$

Projecting to the first component, and with U_0 and U_1 as before, we obtain

$$u(t) = U_0(t - t_0)u_0 + U_1(t - t_0)u_1 + \int_{t_0}^{t} U_1(t - s) f(s) ds.$$

1.54. First, we take a derivative of $\|u(t)\|^2$ in order to obtain the inequality

$$\partial_t \|u(t)\|^2 = 2\langle \partial_t u, u\rangle = 2\langle Lu, u\rangle \leq -2\sigma \|u(t)\|^2.$$

Applying the differential form of the Gronwall inequality, we have

$$\|u(t)\|^2 \leq \|u(0)\|^2 \exp\left(-2 \int_0^t \sigma ds\right) = e^{-2\sigma t} \|u(0)\|^2.$$

The desired inequality follows immediately.

1.57. *Correction:* In contrast to the problem statement, the ODE for ϕ we wish to solve is

$$\partial_t \phi(t) = -P(u(t))\phi(t), \qquad \phi(t_0) = \phi_0.$$

In particular, note the change in sign in the right-hand side of the ODE.

First, a solution ϕ exists, as it can be given explicitly using matrix exponentials:

$$\phi(t) = \exp\left[- \int_{t_0}^{t} P(u(s)) ds\right] \cdot \phi_0.$$

This solution is also unique, since if ϕ and ψ are both solutions, with the same initial condition ϕ_0 at time t_0, then their difference $\alpha = \psi - \phi$ satisfies

$$\partial_t \alpha(t) = -P(u(t))\alpha(t), \qquad \alpha(t_0) = 0.$$

From this, we immediately obtain the estimate

$$\partial_t |\alpha(t)|^2 = -2\langle P(u(t))\alpha(t), \alpha(t)\rangle \leq 2|P(u(t))||\alpha(t)|^2,$$

and Gronwall's inequality ensures that α vanishes for all time.

Consider now the curve

$$t \mapsto v(t) = L(u(t))\phi(t) - \lambda\phi(t)$$

in H, and note first that $v(t_0) = 0$. Differentiating this curve and recalling both the above ODE and the definition of Lax pairs, we have

$$\begin{aligned}\partial_t v(t) &= \partial_t [L(u(t))] \cdot \phi(t) + [L(u(t)) - \lambda] \partial_t \phi(t) \\ &= (LP - PL - LP + \lambda P)|_{u(t)} \phi(t) = -P(u(t)) v(t).\end{aligned}$$

In other words, v satisfies the ODE in the previous paragraph, with vanishing initial data. Therefore, v vanishes everywhere as well by the previous argument. In particular, this shows that the spectrum $\sigma(L)$ of L is an invariant of the flow (1.57) of u.

Chapter 2: Constant Coefficient Linear Dispersive Equations

2.1. Let $z = (z_1, z_2, z_3, z_4) \in V$. The commutation relations are brute force calculations:

$$
\begin{aligned}
\gamma^0\gamma^0 z &= \frac{1}{c^2}(z_1, z_2, z_3, z_4), & \gamma^0\gamma^1 z &= \frac{1}{c}(z_4, z_3, z_2, z_1), \\
\gamma^0\gamma^2 z &= \frac{i}{c}(-z_4, z_3, -z_2, z_1), & \gamma^0\gamma^3 z &= \frac{1}{c}(z_3, -z_4, z_1, -z_2), \\
\gamma^1\gamma^0 z &= \frac{1}{c}(-z_4, -z_3, -z_2, -z_1), & \gamma^1\gamma^1 z &= (-z_1, -z_2, -z_3, -z_4), \\
\gamma^1\gamma^2 z &= i(-z_1, z_2, -z_3, z_4), & \gamma^1\gamma^3 z &= (z_2, -z_1, z_4, -z_3), \\
\gamma^2\gamma^0 z &= -\frac{i}{c}(z_4, -z_3, z_2, -z_1), & \gamma^2\gamma^1 z &= i(z_1, -z_2, z_3, -z_4), \\
\gamma^2\gamma^2 z &= (-z_1, -z_2, -z_3, -z_4), & \gamma^2\gamma^3 z &= i(-z_2, -z_1, -z_4, -z_3), \\
\gamma^3\gamma^0 z &= \frac{1}{c}(-z_3, z_4, -z_1, z_2), & \gamma^3\gamma^1 z &= (-z_2, z_1, -z_4, z_3), \\
\gamma^3\gamma^2 z &= i(z_2, z_1, z_4, z_3), & \gamma^3\gamma^3 z &= (-z_1, -z_2, -z_3, -z_4).
\end{aligned}
$$

As a result, we can check every possibility for $(\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha)z$:

$$
\begin{aligned}
(\gamma^0\gamma^0 + \gamma^0\gamma^0)z &= \frac{2}{c^2}z, & (\gamma^0\gamma^1 + \gamma^1\gamma^0)z &= 0, \\
(\gamma^0\gamma^2 + \gamma^2\gamma^0)z &= 0, & (\gamma^0\gamma^3 + \gamma^3\gamma^0)z &= 0, \\
(\gamma^1\gamma^1 + \gamma^1\gamma^1)z &= -2z, & (\gamma^1\gamma^2 + \gamma^2\gamma^1)z &= 0, \\
(\gamma^1\gamma^3 + \gamma^3\gamma^1)z &= 0, & (\gamma^2\gamma^2 + \gamma^2\gamma^2)z &= -2z, \\
(\gamma^2\gamma^3 + \gamma^3\gamma^2)z &= 0, & (\gamma^3\gamma^3 + \gamma^3\gamma^3)z &= -2z.
\end{aligned}
$$

To see the symmetry of the γ^α's, we let $w = (w_1, w_2, w_3, w_4) \in V$ as well. Then,

$$
\begin{aligned}
\{\gamma^0 z, w\} &= \frac{1}{c}\{(z_1, z_2, -z_3, -z_4), (w_1, w_2, w_3, w_4)\} = \frac{1}{c}(z_1\bar{w}_1 + z_2\bar{w}_2 + z_3\bar{w}_3 + z_4\bar{w}_4), \\
\{z, \gamma^0 w\} &= \frac{1}{c}\{(z_1, z_2, z_3, z_4), (w_1, w_2, -w_3, -w_4)\} = \frac{1}{c}(z_1\bar{w}_1 + z_2\bar{w}_2 + z_3\bar{w}_3 + z_4\bar{w}_4), \\
\{\gamma^1 z, w\} &= \{(z_4, z_3, -z_2, -z_1), (w_1, w_2, w_3, w_4)\} = (z_4\bar{w}_1 + z_3\bar{w}_2 + z_2\bar{w}_3 + z_1\bar{w}_4), \\
\{z, \gamma^1 w\} &= \{(z_1, z_2, z_3, z_4), (w_4, w_3, -w_2, -w_1)\} = (z_1\bar{w}_4 + z_2\bar{w}_3 + z_3\bar{w}_2 + z_4\bar{w}_1), \\
\{\gamma^2 z, w\} &= i\{(-z_4, z_3, z_2, -z_1), (w_1, w_2, w_3, w_4)\} = i(-z_4\bar{w}_1 + z_3\bar{w}_2 - z_2\bar{w}_3 + z_1\bar{w}_4), \\
\{z, \gamma^2 w\} &= -i\{(z_1, z_2, z_3, z_4), (-w_4, w_3, w_2, -w_1)\} = -i(-z_1\bar{w}_4 + z_2\bar{w}_3 - z_3\bar{w}_2 + z_4\bar{w}_1), \\
\{\gamma^3 z, w\} &= \{(z_3, -z_4, -z_1, z_2), (w_1, w_2, w_3, w_4)\} = (z_3\bar{w}_1 - z_4\bar{w}_2 + z_1\bar{w}_3 - z_2\bar{w}_4), \\
\{z, \gamma^3 w\} &= \{(z_1, z_2, z_3, z_4), (w_3, -w_4, -w_1, w_2)\} = (z_1\bar{w}_3 - z_2\bar{w}_4 + z_3\bar{w}_1 - z_4\bar{w}_2).
\end{aligned}
$$

Direct inspection of the above formulas establishes symmetry. In particular, note that

$$\{z, \gamma^0 z\} = c^{-1}|z|^2 \geq 0.$$

To show the final positivity property, we painfully expand expressions. First,

$$
\begin{aligned}
\{z, z\}^2 &= (|z_1|^2 + |z_2|^2 - |z^3|^2 - |z^4|^2)^2 \\
&= |z_1|^4 + |z_2|^4 + |z_3|^4 + |z_4|^4 + 2|z_1|^2|z_2|^2 - 2|z_1|^2|z_3|^2 \\
&\qquad - 2|z_1|^2|z_4|^2 - 2|z_2|^2|z_3|^2 - 2|z_2|^2|z_4|^2 + 2|z_3|^2|z_4|^2.
\end{aligned}
$$

Next, since $\gamma_0 z = c(-z_1, -z_2, z_3, z_4)$, then

$$\begin{aligned}-\{z, \gamma^0 z\}\{z, \gamma_0 z\} &= (|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2)^2, \\ &= |z_1|^4 + |z_2|^4 + |z_3|^4 + |z_4|^4 + 2|z_1|^2|z_2|^2 + 2|z_1|^2|z_3|^2 \\ &\quad + 2|z_1|^2|z_4|^2 + 2|z_2|^2|z_3|^2 + 2|z_2|^2|z_4|^2 + 2|z_3|^2|z_4|^2.\end{aligned}$$

Moreover, since the remaining γ_i's are identical to the γ^i's, then [20]

$$\begin{aligned}-\{z, \gamma^1 z\}\{z, \gamma_1 z\} &= -(z_4\bar{z}_1 + z_3\bar{z}_2 + z_2\bar{z}_3 + z_1\bar{z}_4)^2 = -4[\Re(z_1\bar{z}_4 + z_3\bar{z}_2)]^2 = -A^2, \\ -\{z, \gamma^2 z\}\{z, \gamma_2 z\} &= -(-z_4\bar{z}_1 + z_3\bar{z}_2 - z_2\bar{z}_3 + z_1\bar{z}_4)^2 = -4[\Im(z_1\bar{z}_4 + z_3\bar{z}_2)]^2 = -B^2, \\ -\{z, \gamma^3 z\}\{z, \gamma_3 z\} &= -(z_1\bar{z}_3 - z_2\bar{z}_4 + z_3\bar{z}_1 - z_4\bar{z}_2)^2 = -4[\Re(z_1\bar{z}_3 - z_2\bar{z}_4)]^2 = -C^2,\end{aligned}$$

Combining all the above, then we must show

$$4|z_1|^2|z_4|^2 + 4|z_2|^2|z_3|^2 + 4|z_1|^2|z_3|^2 + 4|z_2|^2|z_4|^2 - A^2 - B^2 - C^2 \geq 0.$$

By a direct calculation while tracking cancellations, then

$$\begin{aligned}-A^2 - B^2 &= -4|z_1\bar{z}_4 + z_3\bar{z}_2|^2 = -4|z_1|^2|z_4|^2 - 4|z_3|^2|z_2|^2 - 8\Re(z_1 z_2 \bar{z}_3 \bar{z}_4), \\ -C^2 &= -4[\Re(z_1\bar{z}_3)]^2 - 4[\Re(z_2\bar{z}_4)]^2 + 8\Re(z_1\bar{z}_3)\Re(z_2\bar{z}_4).\end{aligned}$$

Since

$$\Re(ab) = \Re a \cdot \Re b - \Im a \cdot \Im b$$

for all $a, b \in \mathbb{C}$, then applying Cauchy's inequality yields

$$\begin{aligned}-A^2 - B^2 - C^2 &= -4|z_1|^2|z_4|^2 - 4|z_3|^2|z_2|^2 - 4[\Re(z_1\bar{z}_3)]^2 - 4[\Re(z_2\bar{z}_4)]^2 + 8\Im(z_1\bar{z}_3)\Im(z_2\bar{z}_4) \\ &\geq -4|z_1|^2|z_4|^2 - 4|z_3|^2|z_2|^2 - 4[\Re(z_1\bar{z}_3)]^2 - 4[\Re(z_2\bar{z}_4)]^2 \\ &\qquad - 4[\Im(z_1\bar{z}_3)]^2 - 4[\Im(z_2\bar{z}_4)]^2 \\ &= -4|z_1|^2|z_4|^2 - 4|z_3|^2|z_2|^2 - 4|z_1|^2|z_3|^2 - 4|z_2|^2|z_4|^2.\end{aligned}$$

This completes the proof of the timelike property.

2.2. First, taking a time derivative of the Maxwell equations yields

$$\begin{aligned}\partial_t^2 E &= c^2 \nabla_x \times \partial_t B = -c^2 \nabla_x \times \nabla_x \times E = c^2[\Delta_x E - \nabla_x(\nabla_x \cdot E)] = c^2 \Delta_x E, \\ \partial_t^2 B &= -\nabla_x \times \partial_t E = -c^2 \nabla_x \times \nabla_x \times B = c^2[\Delta_x B - \nabla_x(\nabla_x \cdot B)] = c^2 \Delta B,\end{aligned}$$

Thus, all components of E and B satisfy the wave equation. Next, for the abelian Yang-Mills equations, we take a spacetime divergence of the second equation in (2.6) to obtain

$$0 = \partial^\alpha \partial_\alpha F_{\beta\gamma} + \partial^\alpha \partial_\beta F_{\gamma\alpha} + \partial^\alpha \partial_\gamma F_{\alpha\beta} = \Box F_{\beta\gamma} - \partial_\beta(\partial^\alpha F_{\alpha\gamma}) + \partial_\gamma(\partial^\alpha F_{\alpha\beta}) = \Box F_{\beta\gamma}.$$

In particular, the spacetime divergence of F vanishes due to the first equation in (2.6).

Correction: In order to show that a solution

$$A \in C^2_{t,x}(\mathbb{R}^{1+d} \rightarrow \mathbb{R}^{1+d})$$

of the wave equation can also be reformulated as a solution of the abelian Yang-Mills equations, we must assume in addition the *Lorenz gauge condition*

$$\partial^\alpha A_\alpha \equiv 0.$$

With A as above, we define the "curvature" two-form

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha.$$

[20] Here, $\Re$ and $\Im$ refer to the real and imaginary components, respectively.

A direct computation now yields

$$\partial^\alpha F_{\alpha\beta} = \Box A_\beta - \partial_\beta \partial^\alpha A_\alpha \equiv 0.$$

Furthermore, the definition of F yields the Bianchi identities:

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = \partial_\alpha \partial_\beta A_\gamma - \partial_\alpha \partial_\gamma A_\beta + \partial_\beta \partial_\gamma A_\alpha - \partial_\beta \partial_\alpha A_\gamma + \partial_\gamma \partial_\alpha A_\beta - \partial_\gamma \partial_\beta A_\alpha \equiv 0.$$

Thus, F satisfies the abelian Yang-Mills equations.

We now restrict ourselves to the case $d = 3$. To see that the Maxwell equations are a special case of the abelian Yang-Mills equations, we let F be a solution of the abelian Yang-Mills equations, and we define E and H in terms of F by

$$E_1 = F_{10}, \qquad E_2 = F_{20}, \qquad E_3 = F_{30},$$
$$H_1 = F_{23}, \qquad H_2 = F_{31}, \qquad H_3 = F_{12}.$$

From the first equation in (2.6), we can compute that

$$\begin{aligned} 0 &\equiv \partial^1 F_{10} + \partial^2 F_{20} + \partial^3 F_{30} = \nabla_x \cdot E, \\ 0 &\equiv \partial^0 F_{01} + \partial^2 F_{21} + \partial^3 F_{31} = c^{-2} \partial_t E_1 - (\nabla_x \times H)_1, \\ 0 &\equiv \partial^0 F_{02} + \partial^1 F_{12} + \partial^3 F_{32} = c^{-2} \partial_t E_2 - (\nabla_x \times H)_2, \\ 0 &\equiv \partial^0 F_{03} + \partial^1 F_{13} + \partial^2 F_{23} = c^{-2} \partial_t E_3 - (\nabla_x \times H)_3. \end{aligned}$$

Similarly, for the second equation in (2.6), we can compute

$$\begin{aligned} 0 &\equiv \partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} = \partial_t H_3 + (\nabla_x \times E)_3, \\ 0 &\equiv \partial_0 F_{13} + \partial_1 F_{30} + \partial_3 F_{01} = -\partial_t H_2 - (\nabla_x \times H)_2, \\ 0 &\equiv \partial_0 F_{23} + \partial_2 F_{30} + \partial_3 F_{02} = \partial_t H_1 + (\nabla_x \times E)_1, \\ 0 &\equiv \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = \nabla_x \cdot H. \end{aligned}$$

As a result, we have recovered the Maxwell equations.

Next, if u satisfies the Dirac equations, then we have

$$\frac{m^2 c^2}{\hbar^2} u = i \frac{mc}{\hbar} \gamma^\beta \partial_\beta u = -\gamma^\beta \partial_\beta (\gamma^\alpha \partial_\alpha u) = -\gamma^\alpha \gamma^\beta (\partial_\alpha \partial_\beta u).$$

Since partial derivatives commute, then

$$\frac{m^2 c^2}{\hbar^2} u = -\frac{1}{2} (\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha) \partial_\alpha \partial_\beta u = g^{\alpha\beta} \partial_\alpha \partial_\beta u = \Box u,$$

so u also satisfies the Klein-Gordon equations.

Correction: For the converse, if ϕ satisfies the Klein-Gordon equations, then we show

$$\psi = \gamma^\alpha \partial_\alpha \phi - i \frac{mc}{\hbar} \phi$$

satisfies the Dirac equation. Note the extra factor $-ic\hbar^{-1}$ required on the right-hand side.

With ϕ as above, then a direct computation yields

$$i \gamma^\beta \partial_\beta \psi = -i \Box \phi + \frac{mc}{\hbar} \gamma^\beta \partial_\beta \phi = \frac{mc}{\hbar} \gamma^\beta \partial_\beta \phi - i \frac{m^2 c^2}{\hbar^2} \phi = \frac{mc}{\hbar} \cdot \psi.$$

Thus, ψ indeed satisfies the Dirac equation.

2.5. *Correction:* Define the Galilean transformed function $\tilde{u}$ by [21]

$$E(t,x) = e^{imx\cdot v/\hbar} e^{-imt|v|^2/2\hbar}, \qquad \tilde{u}(t,x) = E(t,x)u(t,x-vt).$$

Direct computations yield

$$\partial_t \tilde{u}(t,x) = E(t,x)\left[\frac{-im|v|^2}{2\hbar} u(t,x-vt) + \partial_t u(t,x-vt) - \sum_j v^j \partial_j u(t,x-vt)\right],$$
$$\partial_j \tilde{u}(t,x) = E(t,x)\left[\frac{imv^j}{\hbar} u(t,x-vt) + \partial_j u(t,x-vt)\right],$$
$$\Delta \tilde{u}(t,x) = E(t,x)\left[\frac{-m^2|v|^2}{\hbar^2} u(t,x-vt) + \Delta u(t,x-vt) + \sum_j \frac{2imv^j}{\hbar} \partial_j u(t,x-vt)\right],$$

Combining the above equations, we obtain

$$\left(i\partial_t + \frac{\hbar}{2m}\Delta\right)\tilde{u}(t,x) = E(t,x)\left(i\partial_t + \frac{\hbar}{2m}\Delta\right)u(t,x-vt).$$

Since $|E| \equiv 1$ (in particular, E is always nonvanishing), it follows that $\tilde{u}$ solves the linear Schrödinger equations if and only if u does.

2.9. Suppose $u : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}$ is a solution of (2.1), and define

$$u_\lambda : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}, \qquad u_\lambda(t,x) = u(\lambda^{-k}t, \lambda^{-1}x),$$

where k is the degree of L and P. Expanding

$$P(\xi) = \sum_{|\alpha|=k} p_\alpha \xi^\alpha, \qquad L = P(\nabla) = \sum_{|\alpha|=k} p_\alpha \partial_x^\alpha, \qquad p_\alpha \in \mathbb{C},$$

then we can compute

$$\partial_t u_\lambda(t,x) = \partial_t[u(\lambda^{-k}t,\lambda^{-1}x)] = \lambda^{-k}\partial_t u(\lambda^{-k}t,\lambda^{-1}x) = \lambda^{-k} Lu(\lambda^{-k}t,\lambda^{-1}x).$$

We can similarly compute Lu_λ via the chain rule:

$$Lu_\lambda(t,x) = \sum_{|\alpha|=k} p_\alpha \partial_x^\alpha [u(\lambda^{-k}t,\lambda^{-1}x)] = \sum_{|\alpha|=k} \lambda^{-k} p_\alpha \partial_x^\alpha u(\lambda^{-k}t,\lambda^{-1}x) = \lambda^{-k} Lu(\lambda^{-k}t,\lambda^{-1}x).$$

The above shows that $\partial_t u_\lambda$ and Lu_λ are the same, so that u_λ solves (2.1).

2.17. Applying the spatial Fourier transform of the transport equation

$$\partial_t u(t,x) = -x_0 \cdot \nabla u(t,x),$$

then we obtain

$$\partial_t \hat{u}(t,\xi) = -i(x_0\cdot\xi)\hat{u}(t,\xi).$$

Solving this ODE with respect to t yields

$$\hat{u}(t,\xi) = e^{-(x_0\cdot\xi)t}\hat{u}_0(\xi) = e^{-(tx_0\cdot\xi)}\hat{u}_0(\xi),$$

and taking an inverse Fourier transform yields

$$u(t,x) = u_0(x - tx_0).$$

As a result,

$$\exp(-tx_0\cdot\nabla)f(x) = f(x - tx_0), \qquad \exp(-x_0\cdot\nabla)f(x) = f(x-x_0).$$

[21]Note the difference in sign in the second exponent.

If f is real analytic, then we can solve for a solution u to the transport equation that is also real analytic in the time variable. We begin by assuming the ansatz

$$u(t,x) = \sum_k a_k(x) \cdot t^k, \qquad u(0,x) = a_0(x) = f(x).$$

The transport equation applied termwise to the summation yields

$$\sum_k (k+1) a_{k+1}(x) \cdot t^k = \sum_k \nabla_{-x_0} a_k(x) \cdot t^k.$$

In other words, for each $k \geq 0$, we must solve

$$a_{k+1}(x) = \frac{1}{k+1} \nabla_{-x_0} a_k(x).$$

Since $a_0 = f$, then by induction, we can show that [22]

$$a_k(x) = \frac{1}{k!} (\nabla_{-x_0})^k f(x) = \frac{1}{k!} (\nabla_{-\frac{x_0}{|x_0|}})^k f(x) \cdot |x_0|^k, \qquad k \geq 0.$$

Plugging this in and recalling Taylor's formula, since f is real analytic, we obtain

$$u(t,x) = \sum_k \frac{1}{k!} (\nabla_{-\frac{x_0}{|x_0|}})^k f(x) \cdot (|x_0| t)^k = f(x - x_0 t).$$

2.18. The spatial Fourier transform of the wave equation is

$$\partial_t \hat{u}(t,\xi) = -|\xi|^2 \hat{u}(t,\xi),$$

which, as an ODE in t, has general solution

$$\hat{u}(t,\xi) = a(\xi) \cos(t|\xi|) + b(\xi) \sin(t|\xi|).$$

Setting $t = 0$ yields $\hat{u}_0(\xi) = a(\xi)$. Next, differentiating the above in time yields

$$\partial_t \hat{u}(t,\xi) = -a(\xi)|\xi| \sin(t|\xi|) + b(\xi)|\xi| \cos(t|\xi|).$$

Setting $t = 0$ yields $\hat{u}_1(\xi) = |\xi| b(\xi)$. Thus, the wave equation has a solution

$$\hat{u}(t,\xi) = \cos(t|\xi|) \cdot \hat{u}_0(\xi) + \frac{\sin(t|\xi|)}{|\xi|} \cdot \hat{u}_1(\xi).$$

Since the operator $-\Delta$ in physical space corresponds to multiplying by the factor $|\xi|^2$ in Fourier space, then $\sqrt{-\Delta}$ corresponds to multiplication by $|\xi|$. Thus, in terms of Fourier multipliers, we can write the above solution in physical space as

$$u(t) = \cos(t\sqrt{-\Delta}) \cdot u_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} \cdot u_1.$$

For the spacetime Fourier transform, we take a Fourier transform in time of the above representation formula for $\hat{u}$. Recalling the standard formulas for the Fourier transforms of the functions $t \mapsto \cos(at)$ and $t \mapsto \sin(at)$, we have

$$\tilde{u}(\tau,\xi) = \pi[\delta(\tau - |\xi|) + \delta(\tau + |\xi|)] \cdot \hat{u}_0(\xi) + \frac{\pi}{i|\xi|}[\delta(\tau - |\xi|) - \delta(\tau + |\xi|)] \cdot \hat{u}_1(\xi).$$

Somewhat informally, since $\delta(\tau - |\xi|)$, $\delta(\tau + |\xi|)$, and $\delta(|\tau| - |\xi|)$ correspond to integrals over the upper null cone, the lower null cone, and the full null cone beginning at the origin, respectively, then one can derive the identities

$$\delta(\tau - |\xi|) + \delta(\tau + |\xi|) = \delta(|\tau| - |\xi|), \qquad \delta(\tau - |\xi|) - \delta(\tau + |\xi|) = \delta(|\tau| - |\xi|) \operatorname{sgn} \tau.$$

[22] In the special case $x_0 = 0$, then solving the above equations for the a_k's yields that $a_k \equiv 0$ for all $k > 0$, so that $u(t,x) = f(x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$, as desired.

For the second formula, one notes that the hyperplanes $\{\tau - |\xi| = 0\}$ and $\{\tau + |\xi| = 0\}$ correspond to the portions of $\{|\tau| - |\xi| = 0\}$ with $\tau > 0$ and $\tau < 0$, respectively. As a result, we obtain the desired formula

$$\tilde{u}(\tau, \xi) = 2\pi \cdot \delta(|\tau| - |\xi|) \left[\frac{1}{2} \hat{u}_0(\xi) + \frac{\operatorname{sgn}(\tau)}{2i|\xi|} \hat{u}_1(\xi) \right].$$

Correction: The above spacetime Fourier identity for the wave equation differs from the problem statement by a factor of 2π.

For the H^s-estimates, we use the Plancherel theorem and the above Fourier identity:

$$\|\nabla u(t)\|_{H^{s-1}_x} \lesssim \|(1 + |\xi|^2)^{\frac{s-1}{2}} |\xi| |\hat{u}_0|^2\|_{L^2_\xi} + \|(1 + |\xi|^2)^{\frac{s-1}{2}} |\hat{u}_1|^2\|_{L^2_\xi} \lesssim \|u_0\|_{H^s_x} + \|u_1\|_{H^{s-1}_x}.$$

Recall that $\cos(t|\xi|)$ and $\sin(t|\xi|)$ are uniformly bounded by 1. Repeating this process with the Fourier identity for $\partial_t \hat{u}$ (and replacing a and b as before), then we obtain the bound

$$\|\partial_t u(t)\|_{H^{s-1}_x} \lesssim \|(1 + |\xi|^2)^{\frac{s-1}{2}} |\xi| |\hat{u}_0|^2\|_{L^2_\xi} + \|(1 + |\xi|^2)^{\frac{s-1}{2}} |\hat{u}_1|^2\|_{L^2_\xi} \lesssim \|u_0\|_{H^s_x} + \|u_1\|_{H^{s-1}_x}.$$

For the lower order bounds, again using the spatial Fourier representations, we obtain

$$\begin{aligned} \|u(t)\|_{H^s_x} &\lesssim \|\nabla u(t)\|_{H^{s-1}_x} + \|u(t)\|_{H^{s-1}_x} \\ &\lesssim \|u_0\|_{H^s_x} + \|u_1\|_{H^{s-1}_x} + \|(1 + |\xi|^2)^{\frac{s-1}{2}} \hat{u}_0\|_{L^2_\xi} + \|(1 + |\xi|^2)^{\frac{s-1}{2}} |\xi|^{-1} \sin(t|\xi|) \hat{u}_1\|_{L^2_\xi} \\ &\lesssim \|u_0\|_{H^s_x} + \|u_1\|_{H^{s-1}_x} + \|(1 + |\xi|^2)^{\frac{s-1}{2}} |\xi|^{-1} \sin(t|\xi|) \hat{u}_1\|_{L^2_\xi}. \end{aligned}$$

If we write the sine factor as

$$|\sin(t|\xi|)| \lesssim \int_0^t |\xi| |\sin(s|\xi|)| ds \lesssim t|\xi|,$$

then we obtain the following control:

$$\begin{aligned} \|u(t)\|_{H^s_x} &\lesssim \|u_0\|_{H^s_x} + \|u_1\|_{H^{s-1}_x} + t\|(1 + |\xi|^2)^{\frac{s-1}{2}} \hat{u}_1\|_{L^2_\xi} \\ &\lesssim \langle t \rangle (\|u_0\|_{H^s_x} + \|u_1\|_{H^{s-1}_x}). \end{aligned}$$

This proves the last identity in the problem statement.

2.21. Define the quantity

$$v(t) = e^{(t_0 - t)L} u(t),$$

and note that

$$\partial_t v(t) = -L e^{(t_0 - t)L} u(t) + e^{(t_0 - t)L} \partial_t u(t) = e^{(t_0 - t)L} F(t).$$

Integrating the above and then applying the propagator $\exp[(t - t_0)L]$ yields

$$e^{(t_0 - t)L} u(t) = u(t_0) + \int_{t_0}^t e^{(t_0 - s)L} F(s) ds, \qquad u(t) = e^{(t - t_0)L} u_0 + \int_{t_0}^t e^{(t - s)L} F(s) ds.$$

2.25. Let $I = [a, b]$. Taking the spatial Fourier transform of the wave equation for u yields

$$\partial_t^2 \hat{u}(t, \xi) = -|\xi|^2 \hat{u}(t, \xi).$$

By the Plancherel theorem,

$$\left\| \int_I u(t, x) dt \right\|_{\dot{H}^2_x} \simeq \left\| \int_I |\xi|^2 \hat{u}(t, \xi) dt \right\|_{L^2_\xi}.$$

By the above Fourier wave equation for $\hat{u}$ and the Plancherel theorem, then

$$\left\| \int_I u(t, x) dt \right\|_{\dot{H}^2_x} \simeq \left\| \int_I \partial_t^2 \hat{u}(t, \xi) dt \right\|_{L^2_\xi} \leq \|\partial_t \hat{u}(b, \xi)\|_{L^2_\xi} + \|\partial_t u(a, \xi)\|_{L^2_\xi}.$$

Applying the energy estimate for the wave equation (see Exercise (2.17)), we have

$$\left\| \int_I u(t,x)dt \right\|_{\dot{H}^2_x} \lesssim \|\partial_t u(0)\|_{L^2_x} \leq \|u(0)\|_{\dot{H}^1_x} + \|\partial_t u(0)\|_{L^2_x}.$$

2.29. [23] Let $I = [a,b]$. For the first estimate, we integrate by parts:

$$\begin{aligned} \int_I e^{i\phi(x)}dx &= \int_a^b \frac{1}{i\phi'(x)} \partial_x [e^{i\phi(x)}] dx \\ &= \frac{1}{i\phi'(b)} e^{i\phi(b)} - \frac{1}{i\phi'(a)} e^{i\phi(a)} + \int_a^b \partial_x \left[\frac{1}{\phi'(x)} \right] e^{i\phi(x)} dx \\ &= I_1 + I_2 + I_3. \end{aligned}$$

By the assumption $|\phi'| \geq \lambda$, we see that $1/\phi'$ has constant sign throughout I, so that

$$|I_1 + I_2| \leq \pm \left[\frac{1}{\phi'(b)} + \frac{1}{\phi'(a)} \right],$$

for the correctly chosen sign. Next, since ϕ'' has constant sign, then

$$|I_3| \leq \pm \int_a^b \partial_x \left[\frac{1}{\phi'(x)} \right] dx = \pm \left[\frac{1}{\phi'(b)} - \frac{1}{\phi'(a)} \right],$$

again for the correctly closen sign. Adding the estimate for $|I_1 + I_2|$ to that for $|I_3|$, we see that one pair of terms have to cancel, so that only one pair remains. As a result,

$$\left| \int_I e^{i\phi(x)}dx \right| \leq 2 \max \left[\frac{1}{|\phi'(b)|}, \frac{1}{|\phi'(a)|} \right] \leq \frac{2}{\lambda}.$$

For general $k > 1$, we use an induction argument. First, the base case $k = 1$ is proved by the above. Suppose now that the desired estimate

$$\left| \int_I e^{i\phi(x)}dx \right| \lesssim_k \lambda^{-\frac{1}{k}}, \qquad \phi \in C^2(I), \qquad |\partial_x^k \phi| \geq \lambda$$

holds for the case k, and consider the case $k+1$. [24]

Let $\phi \in C^{k+1}(I)$ satisfy $|\partial_x^{k+1}\phi| \geq \lambda$, and fix $\delta > 0$. By this assumption on $\partial_x^{k+1}\phi = \partial_x \partial_x^k \phi$, then the uniform lower bound $|\partial_x^k \phi| \geq \delta\lambda$ must hold everywhere on I except possibly for a subinterval I_0 of length at most 2δ. Partition $I \setminus I_0$ into subintervals

$$I^+ = \{x \in I \mid x > y \text{ for all } y \in I_0\}, \qquad I^- = \{x \in I \mid x < y \text{ for all } y \in I_0\}.$$

Applying the induction hypothesis, we have [25]

$$\left| \int_{I^-} e^{i\phi(x)}dx \right| + \left| \int_{I^+} e^{i\phi(x)}dx \right| \lesssim_k (\delta\lambda)^{-\frac{1}{k}} + (\delta\lambda)^{-\frac{1}{k}} \lesssim (\delta\lambda)^{-\frac{1}{k}}.$$

Next, for I_0, we have the trivial estimate

$$\left| \int_{I_0} e^{i\phi(x)}dx \right| \leq 2\delta.$$

As a result, we have

$$\left| \int_I e^{i\phi(x)}dx \right| \lesssim_k (\delta\lambda)^{-\frac{1}{k}} + \delta.$$

[23]The solution was obtained partially from [4].

[24]In the case $k = 1$, we must also assume that ϕ is convex or concave.

[25]Note that if $k = 1$, then our assumption $|\partial_x^{1+1}\phi| \geq \lambda$ automatically implies that ϕ is convex or concave.

Optimizing the inequality by choosing $\delta \sim \lambda^{-1/(k+1)}$, then we obtain as desired

$$\left| \int_I e^{i\phi(x)} dx \right| \lesssim_{k+1} \lambda^{-\frac{1}{k+1}}.$$

Finally, if ψ is a function on I of bounded variation (so that it is differentiable a.e.), then

$$\begin{aligned}
\left| \int_I e^{i\phi(x)} \psi(x) dx \right| &= \left| \int_I \partial_x \int_a^x e^{i\phi(y)} dy \cdot \psi(x) dx \right| \\
&= \left| \int_a^b e^{i\phi(y)} dy \cdot \psi(b) - \int_I \int_a^x e^{i\phi(y)} dy \cdot \psi'(x) dx \right| \\
&\lesssim_k \lambda^{-\frac{1}{k}} \left[|\psi(b)| + \int_I |\psi'(x)| dx \right].
\end{aligned}$$

where in the last step, we applied the previous estimates for the integral of $e^{i\phi(x)}$.

2.35. Suppose the given estimate holds for all $u \in \mathcal{S}_x(\mathbb{R}^d)$, with C_t being the optimal constant (i.e., the operator norm) for a given t. Given $u_0 \in \mathcal{S}_x$ and $\lambda > 0$, we define

$$u_0^\lambda \in \mathcal{S}_x, \qquad u_0^\lambda(x) = u_0(\lambda^{-1} x).$$

By a change of variables, then we obtain the first inequality

$$\| e^{it\Delta/2} u_0^\lambda \|_{L_x^q} \leq C_t t^\alpha \| u_0^\lambda \|_{L_x^p} = C_t t^\alpha \lambda^{\frac{d}{p}} \| u_0 \|_{L_x^p}.$$

The rescaling property of Exercise (2.9) implies the identity

$$e^{it\Delta/2} u_0^\lambda(x) = e^{i \cdot \lambda^{-2} t \cdot \Delta/2} u_0(x/\lambda),$$

so that by a similar change of variables, we have the second inequality

$$\| e^{it\Delta/2} u_0^\lambda \|_{L_x^q} = \lambda^{\frac{d}{q}} \| e^{i \cdot \lambda^{-2} t \cdot \Delta/2} u_0 \|_{L_x^q} \leq C_{\lambda^{-2} t} t^\alpha \lambda^{\frac{d}{q} - 2\alpha} \| u_0 \|_{L_x^p}.$$

By choosing u_0 that almost fulfills the constant C_t in the first inequality above, then dividing the second inequality by the first yields

$$\lambda^{\frac{d}{q} - \frac{d}{q} - 2\alpha} \gtrsim 1.$$

By reversing the roles of the above inequalities, we also have

$$\lambda^{\frac{d}{p} + 2\alpha - \frac{d}{q}} \gtrsim 1.$$

By varying λ over all positive real numbers, it is clear then that

$$\frac{d}{q} - \frac{d}{p} - 2\alpha = 0, \qquad \alpha = \frac{d}{2} \left(\frac{1}{q} - \frac{1}{p} \right).$$

Next, combining the above and Exercise (2.34), we have for any $u_0 \in \mathcal{S}_x(\mathbb{R}^d)$ that

$$\| e^{it\Delta/2} u_0 \|_{L_x^q} \leq C_t t^{\frac{d}{2}(\frac{1}{q} - \frac{1}{p})} \| u_0 \|_{L_x^p}, \qquad \| e^{it\Delta/2} u_0 \|_{L_x^q} \simeq_{u_0} \langle t \rangle^{d(\frac{1}{q} - \frac{1}{2})}.$$

Choose u_0 such that the constant C in the first inequality is almost realized. Dividing each inequality by the other, as before, and varying t over large positive real numbers, we obtain

$$d \left(\frac{1}{q} - \frac{1}{2} \right) = \frac{d}{2} \left(\frac{1}{q} - \frac{1}{p} \right),$$

and by simple algebra this yields $q = p'$.

Finally, if $q < p$, then

$$\frac{d}{2} \left(\frac{1}{q} - \frac{1}{p} \right) > 0,$$

so that for any u_0,

$$\lim_{t\searrow 0}\|e^{it\Delta/2}u_0\|_{L^q_x}\lesssim_{u_0}\lim_{t\searrow 0}t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}=0.$$

However, this contradicts Exercise (2.34), which implies that

$$\|e^{it\Delta/2}u_0\|_{L^q_x}\simeq_{u_0}\langle t\rangle^{d(\frac{1}{q}-\frac{1}{2})},\qquad \lim_{t\searrow 0}\langle t\rangle^{d(\frac{1}{q}-\frac{1}{2})}=1.$$

As a result, we have that $q\geq p$.

2.42. Suppose that the Strichartz inequality (2.24) holds for some p,q,d. If u solves the linear Schrödinger equation, and if $\lambda>0$, then

$$u_\lambda(t,x)=u(\lambda^{-2}t,\lambda^{-1}x)$$

is also a solution of the linear Schrödinger equation, so that

$$\|u_\lambda\|_{L^q_tL^r_x}\lesssim\|u_\lambda(0)\|_{L^2_x},\qquad \lambda>0.$$

By a simple change of variables, we see that

$$\|u_\lambda\|_{L^q_tL^r_x}=\lambda^{\frac{d}{r}+\frac{2}{q}}\|u\|_{L^q_tL^r_x},\qquad \|u_\lambda(0)\|_{L^2_x}=\lambda^{\frac{d}{2}}\|u(0)\|_{L^2_x}.$$

This shows that

$$\lambda^{\frac{d}{r}+\frac{2}{q}-\frac{d}{2}}\|u\|_{L^q_tL^r_x}\lesssim\|u(0)\|_{L^2_x},\qquad \lambda>0,$$

independently of λ. This can only hold if

$$\frac{d}{r}+\frac{2}{q}=\frac{d}{2}.$$

Next, let u be a Schwartz solution of the linear Schrödinger equation, and fix a sequence

$$t_1<t_2<t_3<\dots,$$

with the t_n's spaced "sufficiently far" apart. Moreover, for any integer $N>0$, we define

$$u_N(t)=\sum_{n=1}^N u(t-t_n),$$

which also solves the linear Schrödinger equation. We can estimate $u_N(0)$ in L^2 as follows:

$$\begin{aligned}\|u_N(0)\|^2_{L^2_x}&=\sum_{i=1}^N\|e^{-it_i\Delta}u(0)\|^2_{L^2_x}+\sum_{i\neq j}\langle e^{-it_i\Delta}u(0),e^{-it_j\Delta}u(0)\rangle\\&=N\|u(0)\|^2_{L^2_x}+\sum_{i\neq j}\int_{\mathbb{R}^d}e^{i(t_j-t_i)|\xi|^2}|\hat{u}(0,\xi)|^2d\xi.\end{aligned}$$

Via stationary phase methods, if the t_n's are spaced sufficiently far apart, then

$$\|u_N(0)\|^2_{L^2_x}\lesssim_{u(0)}N+\sum_{i=1}^N\sum_{j=1}^\infty\int_{\mathbb{R}^d}e^{i(t_j-t_i)|\xi|^2}|\hat{u}(0,\xi)|^2d\xi\lesssim_{u(0)}N.$$

As a result, we have obtained $\|u_N(0)\|_{L^2_x}\lesssim_{u(0)}N^{\frac{1}{2}}$.

Next, we consider the $L^q_tL^r_x$-norm of u_N. Let B_n denote the interval $(t_n-\epsilon,t_n+\epsilon)$ for some small enough ϵ so that each B_n is very far away from all the other t_i's. By our assumptions on q and r, then by Exercise (2.34),

$$\|u(t-t_n)\|_{L^r}\simeq_{u,d,r}\langle t-t_n\rangle^{-\frac{2}{q}}.$$

In particular, since the t_n's are spaced very far apart, then for any on $t \in B_n$, we have

$$\|u(t-t_n)\|_{L^r_x} \gg \sum_{\substack{1\leq i\leq\infty \\ i\neq n}} \|u(t-t_i)\|_{L^r_x}, \qquad \|u_N(t)\|_{L^r} \gtrsim \|u(t-t_n)\|_{L^r_x}.$$

Therefore, we can estimate from below as follows:

$$\begin{aligned}\|u_N\|_{L^q_t L^r_x} &\geq \left(\sum_{n=1}^N \int_{B_n} \|u_N(t)\|^q_{L^r_x} dt\right)^{\frac{1}{q}} \\ &\gtrsim \left(\sum_{n=1}^N \int_{B_n} \|u(t-t_n)\|^q_{L^r_x} dt\right)^{\frac{1}{q}} \\ &= \left(N\int_{-\epsilon}^{\epsilon} \|u(t)\|_{L^r_x} dt\right)^{\frac{1}{q}} \\ &\gtrsim N^{\frac{1}{q}}.\end{aligned}$$

In particular, if the Strichartz estimates hold for the above values of q, r, and d, then $q \geq 2$, since otherwise, taking $N \nearrow \infty$, we see that the lower bound for $\|u_N(0)\|_{L^2_x}$ grows faster than the upper bound for $\|u_N\|_{L^q_t L^r_x}$.

2.46. First of all, the Littlewood-Paley and integral Minkowski inequalities imply that

$$\|u(t)\|_{L^r_x} \simeq_{r,d} \left\|\left(\sum_N |P_N u(t)|^2\right)^{\frac{1}{2}}\right\|_{L^r_x} \leq \left(\sum_N \|P_N u(t)\|^2_{L^r_x}\right)^{\frac{1}{2}}$$

for any t. Therefore, applying Minkowski's inequality again, we obtain

$$\begin{aligned}\|u\|_{L^q_t L^r_x(I\times\mathbb{R}^d)} &\lesssim_{r,d} \left[\int_I \left(\sum_N \|P_N u(t)\|^2_{L^r_x}\right)^{\frac{q}{2}} dt\right]^{\frac{1}{q}} \\ &\leq \left[\sum_N \left(\int_I \|P_N u(t)\|^q_{L^r_x} dt\right)^{\frac{2}{q}}\right]^{\frac{1}{2}} \\ &= \left(\sum_N \|P_N u\|^2_{L^q_t L^r_x(I\times\mathbb{R}^d)}\right)^{\frac{1}{2}}.\end{aligned}$$

For the "dual" estimate, we again apply the integral Minkowski inequality to obtain

$$\left(\sum_N \|P_N u\|^2_{L^{q'}_t L^{r'}_x(I\times\mathbb{R}^d)}\right)^{\frac{1}{2}} \leq \left[\int_I \left(\sum_N \|P_N u\|^2_{L^{r'}_x}\right)^{\frac{q'}{2}} dt\right]^{\frac{1}{q'}} \leq \left\|\left(\sum_N |P_N u|^2\right)^{\frac{1}{2}}\right\|_{L^{q'}_t L^{r'}_x(I\times\mathbb{R}^d)}.$$

Another application of the Littlewood-Paley inequality yields the desired dual inequality.

Finally, for the Besov Strichartz inequality, we apply Theorem (2.3) to obtain

$$\left(\sum_N \|P_N e^{it\Delta/2} u_0\|^2_{L^q_t L^r_x}\right)^{\frac{1}{2}} = \left(\sum_N \|e^{it\Delta/2} P_N u_0\|^2_{L^q_t L^r_x}\right)^{\frac{1}{2}} \lesssim \left(\sum_N \|P_N u_0\|^2_{L^2_x}\right)^{\frac{1}{2}} \lesssim \|u_0\|_{L^2_x},$$

since each P_N commutes with $i\partial_t + \Delta$, and hence its linear propagator. Similarly, we have

$$\left\| \int_{\mathbb{R}} e^{-is\Delta/2} F(s) ds \right\|_{L^2_x} \lesssim \left(\sum_N \left\| \int_{\mathbb{R}} e^{-is\Delta/2} P_N F(s) ds \right\|^2_{L^2_x} \right)^{\frac{1}{2}} \lesssim_{d,\tilde{q}',\tilde{r}'} \left(\sum_N \| P_N F(s) \|^2_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x} \right)^{\frac{1}{2}},$$

where in the last step, we applied (2.25). Finally, for the Besov inhomogeneous Strichartz estimate, we apply (2.26) in order to obtain for any band N that

$$\left\| \int_{s<t} e^{i(t-s)\Delta/2} P_N F(s) ds \right\|_{L^q_t L^r_x} \lesssim_{d,q,r,\tilde{q}',\tilde{r}'} \| P_N F(s) \|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}.$$

Taking an ℓ^2-summation of the above over N yields the Besov analogue of (2.26).

2.47. First, we compute the symplectic gradient $\nabla_\omega H$. Since for any "nice" $u, v \in L^2(\mathbb{R}^d)$,

$$\frac{d}{d\varepsilon} H(u + \varepsilon v)|_{\varepsilon=0} = \frac{1}{2} \frac{d}{d\varepsilon} \int_{\mathbb{R}^d} |\nabla u + \varepsilon \nabla v|^2 \Big|_{\varepsilon=0} = \int_{\mathbb{R}^d} \mathrm{Re}(\nabla u \cdot \overline{\nabla v}) = -\int_{\mathbb{R}^d} \mathrm{Re}(\Delta u \cdot \bar{v}),$$

$$\omega(\nabla_\omega H(u), v) = -2 \int_{\mathbb{R}^d} \mathrm{Im}(\nabla_\omega H(u) \cdot \bar{v}) = 2 \int_{\mathbb{R}^d} \mathrm{Re}(i \nabla_\omega H(u) \cdot \bar{v}),$$

then varying v appropriately, we obtain

$$2 \nabla_\omega H(u) = i \Delta u.$$

Thus, Hamilton flow associated to H is the linear Schrödinger equation,

$$i \partial_t u = i \nabla_\omega H(u) = -\frac{1}{2} \Delta u.$$

Since $\{H, H\} \equiv 0$ trivially, then H is conserved on the Hamilton flow of H, i.e.,

$$H(u(t)) = H(u(0)), \qquad i\partial_t u = -\frac{1}{2} \Delta u.$$

This is the conservation of energy. The corresponding symmetry from Noether's theorem is given by by flow along the Hamiltonian equation for H. Since integrating along this flow produces time translates of solutions to the H-Hamiltonian (i.e., Schrödinger) equation, then the corresponding symmetries for H are time translations.

Next, consider the total mass/probability

$$M(u) = \int_{\mathbb{R}^d} |u|^2, \qquad u \in L^2(\mathbb{R}^d).$$

Since for any "nice" $u, v \in L^2(\mathbb{R}^d)$, we have

$$\frac{d}{d\varepsilon} M(u + \varepsilon v)|_{\varepsilon=0} = 2 \int_{\mathbb{R}^d} \mathrm{Re}(u \cdot \bar{v}), \qquad \omega(\nabla_\omega M(u), v) = 2 \int_{\mathbb{R}^d} \mathrm{Re}(i \nabla_\omega M(u) \cdot \bar{v}),$$

then we have $\nabla_\omega M(u) = -iu$. The associated M-Hamiltonian equation is

$$\partial_t u(t) = -iu(t),$$

which has solution flows

$$u(t) = e^{-it} u(0).$$

Note that H is clearly conserved along this flow. By Noether's theorem, we have

$$M(u(t)) = M(u(0)), \qquad i\partial_t u = -\frac{1}{2} \Delta u,$$

which is the conservation of mass/probability. Furthermore, the corresponding symmetry for H is given by the solutions of the M-flows, i.e., the phase rotations.

For the momentum functionals

$$p_j(u) = \int_{\mathbb{R}^d} \operatorname{Im}(\partial_j u \cdot \bar{u}),$$

we can similarly compute

$$\frac{d}{d\varepsilon} p_j(u + \varepsilon v)|_{\varepsilon=0} = \int_{\mathbb{R}^d} \operatorname{Im}(\bar{v} \cdot \partial_j u + \bar{u} \cdot \partial_j v) = 2 \int_{\mathbb{R}^d} \operatorname{Im}(\partial_j u \cdot \bar{v}),$$
$$\omega(\nabla_\omega p_j(u), v) = -2 \int_{\mathbb{R}^d} \operatorname{Im}(\nabla_\omega p_j(u) \cdot \bar{v}).$$

Thus, $\nabla_\omega p_j(u) = -\partial_j u$, so the associated Hamiltonian equation is the transport equation

$$\partial_t u = -\partial_j u,$$

which has solution flows

$$u(t, x) = u(0, x - te_j),$$

where e_j is the unit vector pointing in the positive x^j-direction. As H is clearly conserved by these flows, each p_j is conserved by solutions of the Schrödinger equation. The corresponding symmetry for H is given by solutions of the p_j-flows, which are translations in the x_j-direction. By taking each $1 \leq j \leq d$, we obtain symmetry for all spatial translations.

Finally, for the normalized center-of-mass, which are time-dependent Hamiltonians, we must first extend our "phase space" as in Exercise 1.42. Let $\mathcal{D}$ denote our informal "phase space" for the linear Schrödinger equations, on which ω is defined. We define $\bar{\mathcal{D}} = \mathbb{R}^2 \times \mathcal{D}$, and we define the following symplectic form on $\bar{\mathcal{D}}$:

$$\bar{\omega}((a, b, u), (a', b', u')) = ab' - ba' + \omega(u, u').$$

Again, as in Exercise 1.42, we extend H to a Hamiltonian on $\bar{\mathcal{D}}$:

$$\bar{H} \in C^1(\bar{\mathcal{D}} \to \mathbb{R}), \qquad \bar{H}(a, b, u) = H(u) + b.$$

A direct computation yields that

$$\nabla_{\bar{\omega}} \bar{H} = (1, 0, \nabla_\omega H).$$

As a result, a curve $t \mapsto u(t)$ solves the linear Schrödinger equations, with initial value $u(0) = u_0 \in \mathcal{D}$, if and only if for any $b \in \mathbb{R}$, the curve $t \mapsto u_b(t) = (t, b, u(t))$ solves the $\bar{H}$-Hamilton equations, with initial value $u_b(0) = (0, b, u_0)$.

We now define the normalized center-of-mass Hamiltonians on this extended phase space $\bar{\mathcal{D}}$. Given $1 \leq j \leq d$, we define the functions

$$\mathcal{N}_j \in C^1(\bar{\mathcal{D}} \to \mathbb{R}), \qquad \mathcal{N}_j(a, b, u) = \int_{\mathbb{R}^d} x_j |u|^2 dx - a p_j(u).$$

To compute the symplectic gradient of $\mathcal{N}_j$, we first compute

$$\begin{aligned} \frac{d}{d\varepsilon} \mathcal{N}_j(a + \varepsilon a', b + \varepsilon b', u + \varepsilon v)|_{\varepsilon=0} &= 2 \operatorname{Re} \int_{\mathbb{R}^d} x_j u \bar{v} dx - a' p_j(u) - 2a \int_{\mathbb{R}^d} \operatorname{Im}(\partial_j u \cdot \bar{v}) \\ &= 2 \operatorname{Im} \int_{\mathbb{R}^d} i x_j u \bar{v} dx - a' p_j(u) - 2a \int_{\mathbb{R}^d} \operatorname{Im}(\partial_j u \cdot \bar{v}). \end{aligned}$$

Considering the definition of $\bar{\omega}$, then we see that

$$\nabla_{\bar{\omega}} \mathcal{N}_j(a, b, u) = (0, p_j(u), -i x_j u + a \partial_j u).$$

Hence, the Hamilton equations associated with $\mathcal{N}_j$ are

$$\partial_s(a(s), b(s), u(s)) = (0, p_j(u(s)), -i x_j u(s) + a(s) \partial_j u(s)).$$

Now, suppose $(a(s), b(s), u(s))$ is a solution of the above equation. Then, for any $s \in \mathbb{R}$,

$$\begin{aligned}\frac{d}{ds}\bar{H}(a(s), b(s), u(s)) &= \frac{d}{ds}H(u(s)) + p_j(u(s)) \\ &= \int_{\mathbb{R}^d} \operatorname{Re} \frac{d}{ds}\nabla u(s) \cdot \overline{\nabla u(s)} + p_j(u(s)) \\ &= \int_{\mathbb{R}^d} \operatorname{Re}\{\nabla[-ix_j u(s) + a(s)\partial_j u(s)]\overline{\nabla u(s)}\} + p_j(u(s)) \\ &= \int_{\mathbb{R}^d} \operatorname{Im} u(s)\overline{\partial_j u(s)} + \frac{1}{2}a(s)\int_{\mathbb{R}^d} \partial_j |\nabla u(s)|^2 + p_j(u(s)) \\ &= -p_j(u(s)) + p_j(u(s)) \\ &= 0.\end{aligned}$$

Thus, $\bar{H}$ is conserved by the Hamilton flows of $\mathcal{N}_j$, and therefore by Noether's theorem, $\mathcal{N}_j$ is conserved by the solutions of the linear Schrödinger equation. Finally, solving the $\mathcal{N}_j$-Hamilton equation explicitly for $a(s)$ and $u(s)$, we see that

$$a(s) \equiv t_0 \in \mathbb{R}, \qquad u(s) = e^{-\frac{1}{2}it_0 s^2} e^{-ix_j s} u_0(x + t_0 s e_j),$$

where $e_j \in \mathbb{R}^d$ is the unit vector in the positive x_j-direction. Since $a(s) \equiv t_0$ corresponds to the time variable, then the above curve $s \mapsto u(s)$ generates the Galilean symmetry indicated in Exercise 2.5, in the special case $v = -se_j$. [26] Combining all the above componentwise Galilean symmetries for $1 \leq j \leq d$ yields the general Galilean symmetry.

2.48. Letting $e_0 = |\nabla u|^2/2$, then

$$\partial_t e_0 = \operatorname{Re} \sum_j \partial_j \partial_t u \cdot \overline{\partial_j u} = \operatorname{Re} \frac{i}{2} \sum_j \partial_j \Delta u \cdot \overline{\partial_j u} = \sum_j \partial_j \left[\operatorname{Re} \frac{i}{2} \Delta u \cdot \overline{\partial_j u}\right] - \operatorname{Re} \frac{i}{2} \sum_j |\Delta u|^2.$$

The second term on the right-hand side of course vanishes, while the first can be written as the divergence of the vector field $\operatorname{Re}(\frac{i}{2}\Delta u \overline{\nabla u})$. This is the desired local conservation law.

2.49. First, from the conservation of the pseudo-conformal energy, we have

$$\|(x + it\nabla)u(t)\|_{L^2_x(B_R)} \leq \|(x + it\nabla)u(t)\|_{L^2_x(\mathbb{R}^d)} = \|xu(0)\|_{L^2_x(\mathbb{R}^d)}.$$

As a result, by the conservation of mass, we have

$$\begin{aligned}\|\nabla u(t)\|_{L^2_x(B_R)} &\leq |t|^{-1}[\|xu(t)\|_{L^2_x(B_R)} + \|xu(0)\|_{L^2_x}] \\ &\leq |t|^{-1}[R\|u(t)\|_{L^2_x(B_R)} + \|xu(0)\|_{L^2_x}] \\ &\lesssim \langle R\rangle |t|^{-1} \|\langle x\rangle u(0)\|_{L^2_x}.\end{aligned}$$

2.50. [27] Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a bump function such that

$$\phi(x) = \begin{cases} 1 & |x| \leq 1, \\ 0 & |x| \geq 2. \end{cases}$$

Define

$$M(t) = \left(\int_{\mathbb{R}^d} \phi^2(x/R)|u(t,x)|^2 dx\right)^{\frac{1}{2}}, \qquad t \in \mathbb{R}.$$

[26] Note the correction at the beginning of Exercise 2.5.

[27] Thanks to Kyle Thompson for a technical observation.

Differentiating the above and recalling the Schrödinger equations, we obtain

$$\begin{aligned}\partial_t M(t) &\lesssim M(t)^{-1}\left|\mathrm{Re}\int_{\mathbb{R}^d}\phi^2(x/R)\cdot i\Delta u(t,x)\cdot\overline{u(t,x)}dx\right|\\ &\lesssim M(t)^{-1}\int_{\mathbb{R}^d}|\nabla_x[\phi(x/R)]||\phi(x/R)u(t,x)||\nabla u(t,x)|dx,\end{aligned}$$

where we have integrated by parts in the last step, noting that when the derivative hits $\overline{u}$, then the integrand is purely imaginary. Applying Hölder's inequality yields

$$\partial_t M(t) \lesssim M(t)^{-1}\|\nabla_x[\phi(x/R)]\|_{L^\infty_x}M(t)E^{\frac{1}{2}} \lesssim R^{-1}E^{\frac{1}{2}}.$$

Integrating the above results in the inequality

$$M(t) \le M(0) + O_d(R^{-1}E^{\frac{1}{2}}|t|).$$

Finally, by the definition of ϕ and M, we have for any $t \neq 0$ that

$$\begin{aligned}\left(\int_{|x|\le R}|u(t,x)|^2dx\right)^{\frac{1}{2}} &\le M(t)\\ &\le M(0) + O_d(R^{-1}E^{\frac{1}{2}}|t|)\\ &\le \left(\int_{|x|\le 2R}|u(0,x)|^2dx\right)^{\frac{1}{2}} + O_d(R^{-1}E^{\frac{1}{2}}|t|).\end{aligned}$$

2.52. First, we expand the $H^{k,k}$-norm using the Plancherel theorem:

$$\begin{aligned}\|e^{\frac{1}{2}it\Delta}f\|_{H^{k,k}_x(\mathbb{R}^d)} &= \sum_{j=0}^k \|\langle x\rangle^j e^{\frac{1}{2}it\Delta}f\|_{H^{k-j}_x(\mathbb{R}^d)}\\ &\simeq \sum_{j=0}^k \|\langle\xi\rangle^{k-j}\langle\nabla\rangle^j(e^{\frac{1}{2}it|\xi|^2}\hat{f})\|_{L^2_x(\mathbb{R}^d)}\\ &\lesssim \sum_{a+b\le k}\|\langle\xi\rangle^a\nabla^b(e^{\frac{1}{2}it|\xi|^2}\hat{f})\|_{L^2_x(\mathbb{R}^d)}.\end{aligned}$$

Note that whenever a derivative hits the exponential factor, one picks up an extra factor of ξ and t. Thus, applying the Leibniz rule and induction to the above yields

$$\|e^{\frac{1}{2}it\Delta}f\|_{H^{k,k}_x(\mathbb{R}^d)} \lesssim \sum_{a+b+c\le k}\|t^c\langle\xi\rangle^{a+c}e^{\frac{1}{2}it|\xi|^2}\nabla^b\hat{f}\|_{L^2_x(\mathbb{R}^d)} \lesssim \langle t\rangle^k\sum_{j=0}^k\sum_{l=0}^j\|\langle\xi\rangle^{k-j}\nabla^l\hat{f}\|_{L^2_x(\mathbb{R}^d)}.$$

Applying the Plancherel theorem, we have, by definition,

$$\|e^{\frac{1}{2}it\Delta}f\|_{H^{k,k}_x(\mathbb{R}^d)} \lesssim \langle t\rangle^k\sum_{j=0}^k\|\langle x\rangle^j f\|_{H^{k-j}_x(\mathbb{R}^d)} \lesssim \langle t\rangle^k\|f\|_{H^{k,k}_x(\mathbb{R}^d)}.$$

2.53. First, note we can assume $\varepsilon > 0$ is small, without loss of generality. We wish to adapt the Morawetz-type argument using the smoothed function $a(x) = \langle x\rangle - \varepsilon\langle x\rangle^{1-\varepsilon}$. From (2.37) and the definition of T_{0j}, we have

$$\begin{aligned}\partial_t\int_{\mathbb{R}^3}\partial^j a(x)\cdot\mathrm{Im}[\overline{u(t,x)}\partial_j u(t,x)]\cdot dx &= \int_{\mathbb{R}^3}\partial^{jk}a(x)\cdot\mathrm{Re}(\partial_j u(t,x)\overline{\partial_k u(t,x)})\cdot dx\\ &\quad -\frac{1}{4}\int_{\mathbb{R}^3}|u(t,x)|^2\Delta^2 a(x)\cdot dx\\ &= I_1 + I_2.\end{aligned}$$

We now compute and estimate the derivatives of a.

$$\partial_j a(x) = x_j \langle x\rangle^{-1} + (1-\varepsilon) x_j \langle x\rangle^{-1-\varepsilon},$$

$$\begin{aligned}\partial_{ij} a(x) &= \langle x\rangle^{-1}[1-\varepsilon(1-\varepsilon^2)\langle x\rangle^{-\varepsilon}]\left(\delta_{ij} - \frac{x_i x_j}{|x|^2}\right) + \langle x\rangle^{-3}[1-\varepsilon(1-\varepsilon^2)\langle x\rangle^{-\varepsilon}]\frac{x_i x_j}{|x|^2} \\ &\quad + \varepsilon^2(1-\varepsilon)\langle x\rangle^{-1-\varepsilon}\delta_{ij} \\ &= A_1 \cdot \left(\delta_{ij} - \frac{x_i x_j}{|x|^2}\right) + A_2 \cdot \frac{x_i x_j}{|x|^2} + \varepsilon^2(1-\varepsilon)\langle x\rangle^{-1-\varepsilon}\delta_{ij}.\end{aligned}$$

In particular, this implies the following estimates:

$$|\partial_j a(x)| \lesssim_\varepsilon 1, \qquad |\partial_{ij} a(x)| \lesssim_\varepsilon \langle x\rangle^{-1} + \langle x\rangle^{-3} + \langle x\rangle^{-1-\varepsilon} \lesssim \langle x\rangle^{-1}.$$

Furthermore, we can compute

$$\begin{aligned}\Delta^2 a(x) &= -\varepsilon^2(1+\varepsilon)(1-\varepsilon)(2-\varepsilon)\langle x\rangle^{-3-\varepsilon} \\ &\quad - \langle x\rangle^{-5}\{1\cdot 3\cdot(2\cdot 1-1) - \varepsilon(1+\varepsilon)(3+\varepsilon)[(2+\varepsilon)(1-\varepsilon)-1]\langle x\rangle^{-\varepsilon}\} \\ &\quad - \langle x\rangle^{-7}[1\cdot 1\cdot 3\cdot 5 - \varepsilon(1-\varepsilon)(1+\varepsilon)(3+\varepsilon)(5+\varepsilon)\langle x\rangle^{-\varepsilon}] \\ &= -\varepsilon^2(1+\varepsilon)(1-\varepsilon)(2-\varepsilon)\langle x\rangle^{-3-\varepsilon} - B_1 - B_2.\end{aligned}$$

From our computations for $\partial^2 a$, we can now evaluate I_1. Note first that since ε is sufficiently small, then the factors A_1 and A_2 are both everywhere nonnegative. As a result,

$$\begin{aligned}I_1 &= \int_{\mathbb{R}^3} A_1 \cdot |\nabla\!\!\!/ u(t,x)|^2 \cdot dx + \int_{\mathbb{R}^3} A_2 \cdot |\partial_r u(t,x)|^2 \cdot dx + C_\varepsilon \int_{\mathbb{R}^3} \langle x\rangle^{-1-\varepsilon} |\nabla u(t,x)|^2 dx \\ &\geq C_\varepsilon \int_{\mathbb{R}^3} \langle x\rangle^{-1-\varepsilon}|\nabla u(t,x)|^2 dx,\end{aligned}$$

where $C_\varepsilon > 0$ is a constant depending on ε. Similarly, since $B_1, B_2 \geq 0$ everywhere,

$$I_2 = \frac{D_\varepsilon}{4}\int_{\mathbb{R}^3} \langle x\rangle^{-3-\varepsilon}|u(x,t)|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3}(B_1+B_2)|u(x,t)|^2 dx \geq \frac{D_\varepsilon}{4}\int_{\mathbb{R}^3}\langle x\rangle^{-3-\varepsilon}|u(x,t)|^2 dx.$$

Integrating our initial Morawetz-type identity over the time interval $[-T, T]$ yields

$$\begin{aligned}\int_{\mathbb{R}^3} \partial^j a(x) \cdot \operatorname{Im}(\overline{u(t,x)}\partial_j u(t,x)) \cdot dx\Big|_{t=-T}^{t=T} &\geq C_\varepsilon \int_{-T}^{T}\int_{\mathbb{R}^3} \langle x\rangle^{-1-\varepsilon}|\nabla u(t,x)|^2 dx dt \\ &\quad + \frac{D_\varepsilon}{4}\int_{-T}^{T}\int_{\mathbb{R}^3}\langle x\rangle^{-3-\varepsilon}|u(x,t)|^2 dx dt,\end{aligned}$$

where we have applied the above observations and inequalities. Due to our estimates for a along with mass conservation, we can apply the "momentum estimate" of Lemma A.10 to bound the left-hand side by some constant times $\|u(0)\|^2_{\dot{H}^{1/2}}$. Finally, letting $T \nearrow \infty$ yields

$$\int_{\mathbb{R}}\int_{\mathbb{R}^3}\langle x\rangle^{-1-\varepsilon}|\nabla u(t,x)|^2 dx dt + \int_{\mathbb{R}}\int_{\mathbb{R}^3}\langle x\rangle^{-3-\varepsilon}|u(x,t)|^2 dx dt \lesssim_\varepsilon \|u(0)\|^2_{\dot{H}^{\frac{1}{2}}}.$$

2.58. We first compute the symplectic gradient $\nabla_\omega H$. Formally, we have

$$\begin{aligned}\frac{d}{d\varepsilon} H((u_0,u_1) + \varepsilon(v_0,v_1))|_{\varepsilon=0} &= \frac{1}{2}\frac{d}{d\varepsilon}\int_{\mathbb{R}^d}(|\nabla u_0 + \varepsilon\nabla v_0|^2 + |u_1+\varepsilon v_1|^2)|_{\varepsilon=0} \\ &= \int_{\mathbb{R}^d}(\nabla u_0\cdot\nabla v_0 + u_1\cdot v_1) \\ &= \int_{\mathbb{R}^d}(u_1\cdot v_1 - \Delta u_0 \cdot v_0).\end{aligned}$$

In addition, letting $(w_0, w_1) = \nabla_\omega H(u_0, u_1)$, we have

$$\omega((w_0, w_1), (v_0, v_1)) = \int_{\mathbb{R}^d} (w_0 v_1 - w_1 v_0),$$

so that by varying v_0 and v_1, we have $\nabla_\omega H(u_0, u_1) = (u_1, \Delta u_0)$. Consequently, the Hamilton flow associated with H is $\partial_t(u_0, u_1) = (u_1, \Delta u_0)$, or equivalently, the second-order system

$$\partial_t^2 u_0 = \partial_t u_1 = \Delta u_0, \qquad u_0|_{t=t_0} = \phi, \qquad \partial_t u_0|_{t=t_0} = u_1|_{t=t_0} = \psi.$$

2.59. Recalling the identity (2.46) for the stress-energy tensor, we have

$$\partial_t \int_{\mathbb{R}^d} T^{00} = -\int_{\mathbb{R}^d} \partial_i T^{i0} + \mathrm{Re} \int_{\mathbb{R}^d} \partial^0 u \cdot \overline{F} = -\,\mathrm{Re} \int_{\mathbb{R}^d} \partial_t u \cdot \overline{F},$$

on any timeslice $t = \tau$. Integrating the above with respect to the time over the interval $[0, t]$ and recalling the exact form of T^{00} yields

$$\|\nabla u(t)\|_{L_x^2}^2 + \|\partial_t u(t)\|_{L_x^2}^2 \lesssim \|\nabla u_0\|_{L_x^2}^2 + \|u_1\|_{L_x^2}^2 + \int_0^t \int_{\mathbb{R}^d} |\partial_t u||F| dx d\tau.$$

Taking a supremum over all $t \geq 0$ and applying Hölder's inequality, we have

$$\|\nabla u\|_{C_t^0 L_x^2}^2 + \|\partial_t u(t)\|_{C_t^\infty L_x^2}^2 \lesssim \|\nabla u_0\|_{L_x^2}^2 + \|u_1\|_{L_x^2}^2 + \|\partial_t u\|_{L_t^\infty L_x^2} \|F\|_{L_t^1 L_x^2}.$$

Applying a weighted Cauchy inequality to the last term on the right-hand side completes the proof of the energy estimate (2.28) in the case $s = 1$.

For general $s \in \mathbb{R}$, note that the operator $\langle\nabla\rangle^{s-1}$ commutes with all derivatives, and hence $\langle\nabla\rangle^{s-1} u$ also satisfies the wave equation

$$\Box \langle\nabla\rangle^{s-1} u = \langle\nabla\rangle^{s-1} F.$$

Finally, applying the above estimate (the $s = 1$ case), we obtain

$$\begin{aligned}
\|\nabla u\|_{C_t^0 H_x^{s-1}}^2 + \|\partial_t u\|_{C_t^0 H_x^{s-1}}^2 &\lesssim \|\nabla \langle\nabla\rangle^{s-1} u\|_{C_t^0 L_x^2}^2 + \|\partial_t \langle\nabla\rangle^{s-1} u\|_{C_t^0 L_x^2}^2 \\
&\lesssim \|\langle\nabla\rangle^{s-1} F\|_{L_t^1 L_x^2} \\
&\lesssim \|F\|_{L_t^1 H_x^{s-1}}.
\end{aligned}$$

2.60. *Elaboration:* We assume our variation is compact, that is, X has compact support.

Using the given notations, we have

$$\begin{aligned}
0 &= \frac{d}{ds} S(u_s, g_s)\Big|_{s=0} \\
&= \frac{d}{ds} \int_{\mathbb{R}^{1+d}} L(u_s, g_s) dg_s \Big|_{s=0} \\
&= \int_{\mathbb{R}^{1+d}} \frac{\partial L}{\partial u}(u_s, g_s) \frac{d}{ds} u_s \cdot dg_s \Big|_{s=0} + \sum_{\alpha,\beta} \int_{\mathbb{R}^{1+d}} \frac{\partial L}{\partial g^{\alpha\beta}}(u_s, g_s) \frac{d}{ds} g_s^{\alpha\beta} \cdot dg_s \Bigg|_{s=0} \\
&\quad + \int_{\mathbb{R}^{1+d}} L(u_s, g_s) \frac{d}{ds} \sqrt{|\det g_s|}\Big|_{s=0} \\
&= \int_{\mathbb{R}^{1+d}} \frac{\partial L}{\partial u}(u, g) \frac{d}{ds} u_s|_{s=0} \cdot dg + \sum_{\alpha,\beta} \int_{\mathbb{R}^{1+d}} \frac{\partial L}{\partial g^{\alpha\beta}}(u, g) \frac{d}{ds} g_s^{\alpha\beta}|_{s=0} \cdot dg \\
&\quad + \int_{\mathbb{R}^{1+d}} L(u, g) \, \frac{d}{ds} \sqrt{|\det g_s|}\Big|_{s=0} \\
&= A + B + C.
\end{aligned}$$

The term A corresponds to the Euler-Lagrange equation for the fixed-metric Lagrangian $T(u) = S(u, g)$. Since u is a critical point of T by assumption, A vanishes. To evaluate B and C, we must evaluate derivatives of components of the metric. First,

$$\frac{d}{ds} g_s^{\alpha\beta} \Big|_{s=0} = - g_s^{\alpha\mu} g_s^{\beta\nu} \frac{d}{ds} (g_s)_{\mu\nu} \Big|_{s=0} = -g^{\alpha\mu} g^{\beta\nu} \pi_{\mu\nu}.$$

Similarly, since $|\det g_s| = -\det g_s$ (due to the Lorentzian signature), we can compute

$$\frac{d}{ds} \sqrt{|\det g_s|} \Big|_{s=0} = \frac{1}{2} (-\det g)^{-\frac{1}{2}} \frac{d}{ds} (-\det g_s) \Big|_{s=0} = \frac{1}{2} (-\det g)^{\frac{1}{2}} \cdot g^{\mu\nu} \pi_{\mu\nu}.$$

Combining the above observations, we obtain

$$\begin{aligned} 0 &= B + C \\ &= \int_{\mathbb{R}^{1+d}} \left[-g^{\alpha\mu} g^{\beta\nu} \frac{\partial L}{\partial g^{\alpha\beta}} (u, g) + \frac{1}{2} g^{\mu\nu} L(u, g) \right] \pi_{\mu\nu} dg \\ &= - \int_{\mathbb{R}^{1+d}} g^{\mu\alpha} g^{\nu\beta} T_{\alpha\beta} \pi_{\mu\nu} dg \\ &= - \int_{\mathbb{R}^{1+d}} T^{\mu\nu} \pi_{\mu\nu} dg. \end{aligned}$$

This completes the first part of the problem.

Since $T^{\alpha\beta}$ is symmetric in α and β, then

$$T^{\alpha\beta} \pi_{\alpha\beta} = T^{\alpha\beta} (\nabla_\alpha X_\beta + \nabla_\beta X_\alpha) = 2 T^{\alpha\beta} \nabla_\alpha X_\beta.$$

As a result, by the (spacetime) divergence theorem,

$$0 = \int_{\mathbb{R}^{1+d}} T^{\alpha\beta} \nabla_\alpha X_\beta = - \int_{\mathbb{R}^{1+d}} \nabla_\alpha T^{\alpha\beta} \cdot X_\beta.$$

Since this holds for arbitrary X (say, of compact support), then $\nabla_\alpha T^{\alpha\beta} \equiv 0$, i.e., T is divergence-free. Finally, in the special case $L(u, g) = g^{\alpha\beta} \partial_\alpha u \partial_\beta u$, we have

$$T_{\alpha\beta} = \frac{\partial L}{\partial g^{\alpha\beta}} (u, g) - \frac{1}{2} g_{\alpha\beta} L(u, g) = \partial_\alpha u \partial_\beta u - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} \partial_\mu u \partial_\nu u.$$

2.64. Let X denote the radial field

$$\partial_r = \frac{x}{|x|} \cdot \nabla_x.$$

We can compute the deformation tensor π with respect to X, here with respect to Cartesian coordinates. First, since X is time-independent and has no time component, then

$$\pi_{0\beta} = \pi_{\beta 0} \equiv 0, \qquad 0 \leq \beta \leq 3.$$

Next, if i and j are spatial indices, i.e., $1 \leq i, j \leq 3$, then

$$\pi_{ij} = \partial_i X_j + \partial_j X_i = 2 \cdot \frac{|x|^2 \delta_{ij} - x_i x_j}{|x|^3}.$$

Since T is divergence-free, then

$$\begin{aligned} \partial_\alpha (T^{\alpha\beta} X_\beta) &= \frac{1}{2} T^{ij} \pi_{ij} \\ &= (\partial^i u \partial^j u - \frac{1}{2} \delta^{ij} \partial_\alpha u \partial^\alpha u)(|x|^{-1} \delta_{ij} - |x|^{-3} x_i x_j) \\ &= \frac{|\nabla_x u|^2}{|x|} - \frac{3}{2|x|} \partial_\alpha u \partial^\alpha u - \frac{1}{|x|} (\partial_r u)^2 + \frac{1}{2|x|} \partial_\alpha u \partial^\alpha u \end{aligned}$$

$$= \frac{|\nabla\!\!\!/_x u|^2}{|x|} - \frac{1}{|x|}\partial_\alpha u \partial^\alpha u$$
$$= \frac{|\nabla\!\!\!/_x u|^2}{|x|} - \frac{1}{2|x|}\Box(|u|^2).$$

Next, fix a cutoff function η on $\mathbb{R}$ supported on $[-1, 1]$, fix $T_0 > 0$, and define the rescaling $\eta_{T_0}(t) = \eta(t/T_0)$. On one hand, by the spacetime divergence theorem,

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}^3} \eta_{T_0}(t)\cdot\partial_\alpha(T^{\alpha\beta}X_\beta)\cdot dxdt &= \int_{\mathbb{R}}\int_{\mathbb{R}^3} \partial_\alpha[\eta_{T_0}(t)\cdot T^{\alpha\beta}X_\beta|_{(t,x)}]dxdt \\ &\quad - \int_{\mathbb{R}}\int_{\mathbb{R}^3} \partial_t\eta_{T_0}(t)\cdot T^{0\beta}X_\beta\cdot dxdt \\ &= 0 - I_1.\end{aligned}$$

Note in particular that due to the cutoff function η_{T_0} and the rapid decay of u in the spatial directions, the divergence theorem yields no boundary terms. Moreover, since X has unit length everywhere, and since $|T^{\alpha\beta}| \lesssim |D_{x,t}u|^2$, then

$$|I_1| \lesssim \|\partial_t\eta_{T_0}\|_{L^\infty_t}\int_{-T_0}^{T_0}\int_{\mathbb{R}^3}|D_{t,x}u|^2 dxdt \lesssim_\eta \frac{1}{T_0}\int_{-T_0}^{T_0}\int_{\mathbb{R}^3}|D_{t,x}u|^2dxdt \lesssim E,$$

where

$$E = \frac{1}{2}\|u(0)\|_{\dot{H}^1_x} + \frac{1}{2}\|\partial_t u(0)\|_{L^2_x},$$

and where we have applied the standard energy conservation for the wave equation.

On the other hand, we also have that from our expansion for $\partial_\alpha(T^{\alpha\beta})$ that

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\eta_{T_0}(t)\partial_\alpha(T^{\alpha\beta}X_\beta)|_{(t,x)}dxdt &= \int_{\mathbb{R}}\int_{\mathbb{R}^3}\frac{1}{|x|}\eta_{T_0}(t)[|\nabla\!\!\!/_x u|^2 + \partial_t^2(|u|^2) - \Delta(|u|^2)]dxdt \\ &= I_2 + I_3 + I_4.\end{aligned}$$

The term I_2, we can simply leave alone. For I_4, recalling that the distribution $-(4\pi|x|)^{-1}$ is the fundamental solution of Δ (with respect to the origin of $\mathbb{R}^3$), then we can compute [28]

$$I_4 = 4\pi\int_{\mathbb{R}}\eta_{T_0}(t)|u(0,t)|^2dt.$$

Lastly, for I_3, we integrate by parts once and apply Hölder's inequality:

$$|I_3| \lesssim \|\partial_t\eta_{T_0}\|_{L^\infty_t}\int_{-T_0}^{T_0}\int_{\mathbb{R}^3}\frac{|\partial_t(|u|^2)|}{|x|^2}dxdt \lesssim_\eta \frac{1}{T_0}\int_{-T_0}^{T_0}\|\partial_t u(t)\|_{L^2_x}\left(\int_{\mathbb{R}^3}\frac{|u(x,t)|^2}{|x|^2}dx\right)^{\frac{1}{2}}dt.$$

Applying Hardy's inequality (Lemma A.2) to the spatial integral, we have

$$|I_3| \lesssim \frac{1}{T_0}\int_{-T_0}^{T_0}\|\partial_t u(t)\|_{L^2_x}\|\nabla_x u(t)\|_{L^2_x}dt \lesssim E,$$

where we have again applied the conservation of energy.

Combining all the above yields our desired inequality

$$\int_{\mathbb{R}}\int_{\mathbb{R}^3}\eta_{T_0}(t)\frac{|\nabla\!\!\!/_x u(t,x)|^2}{|x|^2}dxdt + \int_{\mathbb{R}}\eta_{T_0}(t)|u(t,x)|^2 \simeq I_2 + I_4 \lesssim E.$$

Moreover, note that the above inequality holds *independently of* T_0. Thus, letting $T_0 \nearrow \infty$ and applying the dominated convergence theorem yields the desired Morawetz estimate.

[28]Of course, one can compute I_4 explicitly, by replacing the spatial integral over $\mathbb{R}^3$ by the same integral over $\mathbb{R}^3$ minus a ball of radius ε about the origin and then letting $\varepsilon \searrow 0$.

2.69. First, we can directly compute

$$\partial_\alpha T^{\alpha\beta} = F\partial^\beta u + \partial^\alpha u \cdot \partial_\alpha \partial^\beta u - \frac{1}{2}\partial^\beta(\partial_\alpha u \cdot \partial^\alpha u) = F\partial^\beta u.$$

Consider the vector field [29]

$$X = e^{2tx_j}\partial_j, \qquad X_k = \delta_{jk} e^{2tx_k}.$$

By the divergence theorem,

$$0 = \int_{\mathbb{R}^d} \partial_\alpha (T^{\alpha\beta} X_\beta) = \int_{\mathbb{R}^d} F\partial^\beta u \cdot X_\beta + \int_{\mathbb{R}^d} T^{\alpha\beta}\partial_\alpha X_\beta = -I_1 + I_2.$$

For I_1, we can expand

$$I_1 = -\int_{\mathbb{R}^d} e^{2tx_j} F\partial_j u = -\int_{\mathbb{R}^d} e^{tx_j} F\partial_j (e^{tx_j} u) + t\int_{\mathbb{R}^d} Fue^{2tx_j}.$$

Next, since $\partial_\alpha X_\beta = \delta_{\alpha j}\delta_{\beta j} 2te^{2tx_j}$, then

$$\begin{aligned} I_2 &= 2t\int_{\mathbb{R}^d} T^{jj} e^{2tx_j} \\ &= 2t\int_{\mathbb{R}^d} |\partial^j u|^2 e^{2tx_j} - t\int_{\mathbb{R}^d} \partial^i u \partial_i u e^{2tx_j} \\ &= 2t\int_{\mathbb{R}^d} e^{tx_j}\partial_j u \partial_j (e^{tx_j} u) - 2t^2 \int_{\mathbb{R}^d} \partial_j u \cdot u \cdot e^{2tx_j} + t\int_{\mathbb{R}^d} uFe^{2tx_j} + 2t^2 \int_{\mathbb{R}^d} u\cdot\partial_j u \cdot e^{2tx_j} \\ &= 2t\int_{\mathbb{R}^d} |\partial_j (e^{tx_j} u)|^2 - 2t^2 \int_{\mathbb{R}^d} e^{tx_j} u \cdot \partial_j (e^{tx_j} u) + t\int_{\mathbb{R}^d} uFe^{2tx_j}. \end{aligned}$$

Since the second term on the right vanishes (by the fundamental theorem of calculus), and since $I_1 = I_2$ by our previous calculations, then

$$2t\int_{\mathbb{R}^d} |\partial_j (e^{tx_j} u)|^2 = -\int_{\mathbb{R}^d} e^{tx_j} F\partial_j (e^{tx_j} u).$$

Finally, applying Hölder's inequality to the above yields the Carleman inequality

$$\|\partial_j (e^{tx_j} u)\|_{L^2} \leq \frac{1}{2|t|}\|e^{tx_j} F\|_{L^2}.$$

Suppose now that $\Delta u = O(|u|)$. Since u is compactly supported, then

$$\frac{1}{2}\|e^{tx_j} u\|_{L^2}^2 = \int_{\mathbb{R}^d}\int_{-\infty}^{x_j} \partial_j (e^{ts} u) \cdot e^{tx_j} u \cdot ds dx \leq \|e^{tx_j} u\|_{L^2} \left[\int_{\mathbb{R}} \int_{\mathbb{R}^d} |\partial_j (e^{ts} u)|^2 dx ds \right]^{\frac{1}{2}}.$$

If R is chosen so that u is supported entirely in the region $\{|x_j| \leq R\}$, then

$$\|e^{tx_j} u\|_{L^2} \leq 2R\|\partial_j (e^{tx_j} u)\|_{L^2} \leq \frac{R}{|t|}\|e^{tx_j} u\|_{L^2},$$

where we applied the Carleman inequality in the last step. Taking t to be sufficiently large forces $\|e^{tx_j} u\|_{L^2} = 0$, which implies $u \equiv 0$ and proves the unique continuation property.

[29] Here, t is simply a nonzero constant.

2.70. *Correction:* The correct identity we wish to show is

$$\|u\|_{X^{s,b}_{\tau=h(\xi)}} = \|v\|_{H^b_t H^s_x}.$$

Let $\mathcal{F}_x$ and $\mathcal{F}_t$ denote Fourier transforms in space and time, respectively. Since

$$u(t) = U(t)v(t) = e^{tL}v(t), \qquad \mathcal{F}_x u(t) = e^{ith(\xi)}\mathcal{F}_x v(t),$$

then we have

$$\begin{aligned}
\|u\|_{X^{s,b}_{\tau=h(\xi)}} &= \|\langle\xi\rangle^s\langle\tau - h(\xi)\rangle^b \cdot \mathcal{F}_t[e^{ith(\xi)}\mathcal{F}_x v(t)]\|_{L^2_\tau L^2_\xi} \\
&= \|\langle\xi\rangle^s\langle\tau - h(\xi)\rangle^b \cdot \mathcal{F}_t\mathcal{F}_x[v(\tau - h(\xi), \xi)]\|_{L^2_\tau L^2_\xi} \\
&= \|\langle\xi\rangle^s\langle\tau\rangle^b \cdot \mathcal{F}_t\mathcal{F}_x[v(\tau, \xi)]\|_{L^2_\tau L^2_\xi} \\
&= \|v\|_{H^b_t H^s_x},
\end{aligned}$$

where we applied the Plancherel theorem in the last step.

2.75. [30] We first prove the analogous estimate for solutions of the linear Schrödinger equation. More specifically, we show that if $u_0, v_0 \in \mathcal{S}_x(\mathbb{R}^d)$, and if $\hat{u}_0$ and $\hat{v}_0$ are supported in the Fourier domains $|\xi| \leq M$ and $|\xi| \geq N$, respectively, then

$$\|e^{it\Delta}u_0 e^{it\Delta}v_0\|_{L^2_t L^2_x} \lesssim_d \frac{M^{\frac{d-1}{2}}}{N^{\frac{1}{2}}}\|u_0\|_{L^2_x}\|v_0\|_{L^2_x}.$$

Again, if $d = 1$, we also require that $N > 2M$.

First, suppose $d \geq 2$ and $N \lesssim_d M$. Then, applying the Gagliardo-Nirenberg inequality (Proposition A.3) and the Strichartz inequality, we obtain, as desired,

$$\begin{aligned}
\|e^{it\Delta}u_0 e^{it\Delta}v_0\|_{L^2_t L^2_x} &\lesssim_d \|e^{it\Delta}u_0\|_{L^4_t L^{2d}_x}\|e^{it\Delta}v_0\|_{L^4_t L^{\frac{2d}{d-1}}_x} \\
&\lesssim_d \||\nabla|^{\frac{d-2}{2}} e^{it\Delta}u_0\|_{L^4_t L^{\frac{2d}{d-1}}_x}\|e^{it\Delta}v_0\|_{L^4_t L^{\frac{2d}{d-1}}_x} \\
&\lesssim_d \||\nabla|^{\frac{d-2}{2}} u_0\|_{L^2_x}\|v_0\|_{L^2_x} \\
&\lesssim M^{\frac{d-2}{2}}\|u_0\|_{L^2_x}\|v_0\|_{L^2_x} \\
&\lesssim_d \frac{M^{\frac{d-1}{2}}}{N^{\frac{1}{2}}}\|u_0\|_{L^2_x}\|v_0\|_{L^2_x}.
\end{aligned}$$

Next, we consider any arbitrary dimension d, but with $N \gg_d M$ (in the case $d = 1$, we need only assume that $N > 2M$). By duality, it suffices to prove

$$I = \left|\int_{\mathbb{R}}\int_{\mathbb{R}^d} e^{it\Delta}u_0(x)e^{it\Delta}v_0(x)F(t,x)dxdt\right| \lesssim_d \frac{M^{\frac{d-1}{2}}}{N^{\frac{1}{2}}}\|u_0\|_{L^2_x}\|v_0\|_{L^2_x}\|F\|_{L^2_t L^2_x}.$$

By Parseval's identity, and recalling that

$$\mathcal{F}_{t,x}(e^{it\Delta}u_0)(\tau,\xi) = \delta(\tau - |\xi|^2)\hat{u}_0(\xi), \qquad \mathcal{F}_{t,x}(e^{it\Delta}v_0)(\tau,\xi) = \delta(\tau - |\xi|^2)\hat{v}_0(\xi),$$

in the distributional sense, we can expand I as follows:

$$\begin{aligned}
I &\simeq \left|\int_{\mathbb{R}}\int_{\mathbb{R}^d} \mathcal{F}_{t,x}(e^{it\Delta}u_0 e^{it\Delta}v_0)(\tau,\xi)\tilde{F}(\tau,\xi)d\xi d\tau\right| \\
&\simeq \left|\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{(\tau_1,\xi_1)+(\tau_2,\xi_2)=(\tau,\xi)} \delta(\tau_1 - |\xi_1|^2)\hat{u}_0(\xi_1)\delta(\tau_2 - |\xi_2|^2)\hat{v}_0(\xi_2)d\xi_1 d\tau_1\tilde{F}(\tau,\xi)d\xi d\tau\right|
\end{aligned}$$

[30]Much of the solution was obtained from [2].

$$\simeq \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \hat{u}_0(\xi_1) \hat{v}_0(\xi_2) \tilde{F}(|\xi_1|^2 + |\xi_2|^2, \xi_1 + \xi_2) d\xi_1 d\xi_2 \right| .$$

Applying Hölder's inequality and recalling the supports of $\hat{u}_0$ and $\hat{v}_0$, then

$$\begin{aligned} I &\lesssim \|u_0\|_{L^2_x} \left\{ \int_{|\xi_1| \leq M} \left[\int_{\mathbb{R}^d} \hat{v}_0(\xi_2) \tilde{F}(|\xi_1|^2 + |\xi_2|^2, \xi_1 + \xi_2) d\xi_2 \right]^2 d\xi_1 \right\}^{\frac{1}{2}} \\ &\lesssim \|u_0\|_{L^2_x} \|v_0\|_{L^2_x} \left[\int_{|\xi_1| \leq M} \int_{|\xi_2| \geq N} |\tilde{F}(|\xi_1|^2 + |\xi_2|^2, \xi_1 + \xi_2)|^2 d\xi_2 d\xi_1 \right]^{\frac{1}{2}} . \end{aligned}$$

Given $1 \leq i \leq d$, we define the domain

$$D_i = \{(\xi_1, \xi_2) \in \mathbb{R}^d \times \mathbb{R}^d \mid |\xi_1| \leq M, |\xi_2| \geq N, |\xi_2^i| \geq N d^{-\frac{1}{2}}\},$$

where ξ_1^i is the i-th component of ξ_1. Note that

$$\bigcup_{i=1}^d D_i = \{(\xi_1, \xi_2) \in \mathbb{R}^d \times \mathbb{R}^d \mid |\xi_1| \geq M, |\xi_2| \geq N\}.$$

Consider the change of variables

$$s = \xi_1 + \xi_2, \qquad r = |\xi_1|^2 + |\xi_2|^2, \qquad \xi_1' = \xi_1',$$

on D_i, where $\xi_1' \in \mathbb{R}^{d-1}$ represents $\xi_1 \in \mathbb{R}^d$ but with the i-th component ξ_1^i omitted. An explicit calculation yields the following value for the corresponding Jacobian:

$$J = \left| \frac{\partial(r, \xi_1', s)}{\partial(\xi_1^i, \xi_1', \xi_2)} \right| = 2|\xi_1^i \pm \xi_2^i|,$$

Here, the sign in "$\pm$" depends on the dimension d. By our assumption $N \gg_d M$ (or $N > 2M$ when $d = 1$) and from our definition of D_i, we have that $J \gtrsim N$ on D_i. Integrating now over D_i and applying this change of variables, we have

$$\begin{aligned} \int_{D_i} |\tilde{F}(|\xi_1|^2 + |\xi_2|^2, \xi_1 + \xi_2)|^2 d\xi_1 d\xi_2 &\lesssim \int_{|\xi_1'| \leq M} d\xi_1' \int_{\mathbb{R}} \int_{\mathbb{R}^d} |\tilde{F}(r, s)|^2 \|J^{-1}\|_{L^\infty(D_i)} ds dr \\ &\lesssim_d M^{d-1} N^{-1} \|F\|^2_{L^2_t L^2_x}. \end{aligned}$$

As a result, combining all the above, we obtain

$$I \lesssim \|u_0\|_{L^2_x} \|v_0\|_{L^2_x} \left[\sum_{i=1}^d \int_{D_i} |\tilde{F}(|\xi_1|^2 + |\xi_2|^2, \xi_1 + \xi_2)|^2 d\xi_2 d\xi_1 \right]^{\frac{1}{2}} \lesssim_d \frac{M^{\frac{d-1}{2}}}{N^{\frac{1}{2}}} \|u_0\|_{L^2_x} \|v_0\|_{L^2_x} \|F\|_{L^2_t L^2_x}.$$

This completes the proof of the case $N \gg_d M$. As a result, we have proved

$$\|e^{it\Delta} u_0 e^{it\Delta} v_0\|_{L^2_t L^2_x} \lesssim_d \frac{M^{\frac{d-1}{2}}}{N^{\frac{1}{2}}} \|u_0\|_{L^2_x} \|v_0\|_{L^2_x},$$

with u_0 and v_0 as before. It remains to convert the above into an $X^{s,b}$-type estimate.

Let $u, v \in \mathcal{S}_{t,x}(\mathbb{R} \times \mathbb{R}^d)$ satisfy the hypotheses in the problem statement. Given any $\sigma, \tau \in \mathbb{R}$, we define the following functions on $\mathbb{R}^d$:

$$f_\sigma(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \tilde{u}(|\xi|^2 + \sigma, \xi) e^{ix\cdot\xi} d\xi, \qquad g_\tau(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \tilde{v}(|\xi|^2 + \tau, \xi) e^{ix\cdot\xi} d\xi.$$

Note in particular that $\hat{f}_\sigma$ and $\hat{g}_\tau$ are supported in the Fourier domains $|\xi| \leq M$ and $|\xi| \geq N$, respectively, for any σ and τ. From the proof of Lemma 2.9, we have the identities

$$u(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{it\sigma} e^{it\Delta} f_\sigma d\sigma, \qquad v(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{it\tau} e^{it\Delta} g_\tau d\tau.$$

Using the above identities, we obtain

$$\begin{aligned} \|uv\|_{L^2_t L^2_x} &\lesssim \left\| \int_{\mathbb{R}} \int_{\mathbb{R}} e^{it\sigma} e^{it\Delta} f_\sigma e^{it\tau} e^{it\Delta} g_\tau d\sigma d\tau \right\|_{L^2_t L^2_x} \\ &\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} \|e^{it\Delta} f_\sigma e^{it\Delta} g_\tau\|_{L^2_t L^2_x} d\sigma d\tau \\ &\lesssim_d \frac{M^{\frac{d-1}{2}}}{N^{\frac{1}{2}}} \int_{\mathbb{R}} \|f_\sigma\|_{L^2_x} d\sigma \int_{\mathbb{R}} \|g_\tau\|_{L^2_x} d\tau, \end{aligned}$$

where in the last step, we applied the free Schrödinger estimate established above. Finally, applying Hölder's inequality as in the proof of Lemma 2.9, we have

$$\|uv\|_{L^2_t L^2_x} \lesssim_{d,b} \frac{M^{\frac{d-1}{2}}}{N^{\frac{1}{2}}} \left[\int_{\mathbb{R}} \langle \sigma \rangle^{2b} \|f_\sigma\|^2_{L^2_x} d\sigma \right]^{\frac{1}{2}} \left[\int_{\mathbb{R}} \langle \tau \rangle^{2b} \|g_\tau\|^2_{L^2_x} d\tau \right]^{\frac{1}{2}} \lesssim \frac{M^{\frac{d-1}{2}}}{N^{\frac{1}{2}}} \|u\|_{X^{0,b}_{\tau=|\xi|^2}} \|v\|_{X^{0,b}_{\tau=|\xi|^2}}.$$

Chapter 3: Semilinar Dispersive Equations

3.1. First, for the NLS, consider the symplectic form ω and the Hamiltonian H, given by

$$\omega(u, v) = -2 \int_{\mathbb{R}^d} \operatorname{Im}(u\bar{v}), \qquad H(u) = \int_{\mathbb{R}^d} \left(\frac{1}{2} |\nabla u|^2 + \frac{2}{p+1} \mu |u|^{p+1} \right),$$

where u and v are in the appropriate spaces. Taking a (directional) derivative of H yields

$$\begin{aligned} \frac{d}{d\varepsilon} H(u + \varepsilon v)|_{\varepsilon=0} &= \frac{d}{d\varepsilon} \int_{\mathbb{R}^d} \left[\frac{1}{2} |\nabla(u + \varepsilon v)|^2 + \frac{2}{p+1} \mu |u + \varepsilon v|^{p+1} \right] \Bigg|_{\varepsilon=0} \\ &= \int_{\mathbb{R}^d} [\operatorname{Re}(\nabla u \cdot \nabla \bar{v}) + 2\mu |u|^{p-1} \operatorname{Re}(u\bar{v})] \\ &= \operatorname{Im} \int_{\mathbb{R}^d} (-i\Delta u + 2i\mu |u|^{p-1} u) \bar{v} \\ &= \omega \left(\frac{1}{2} i\Delta u - i\mu |u|^{p-1} u, v \right). \end{aligned}$$

As a result, the symplectic gradient of H is

$$\nabla_\omega H(u) = \frac{1}{2} i\Delta u - i\mu |u|^{p-1} u,$$

and hence the Hamiltonian evolution equation is

$$\partial_t u = \frac{1}{2} i\Delta u - i\mu |u|^{p-1} u, \qquad i\partial_t u + \frac{1}{2} \Delta u = \mu |u|^{p-1} u.$$

Similarly, for the NLW, we define ω and H, also on appropriate spaces, by

$$\begin{aligned} \omega((u_0, u_1), (v_0, v_1)) &= \int_{\mathbb{R}^d} (u_0 v_1 - v_0 u_1), \\ H(u_0, u_1) &= \int_{\mathbb{R}^d} \left(\frac{1}{2} |\nabla u_0|^2 + \frac{1}{2} |u_1|^2 + \frac{1}{p+1} \mu |u_0|^{p+1} \right). \end{aligned}$$

Again, taking a directional derivative yields

$$\begin{aligned}\frac{d}{d\varepsilon} H(u_0 + \varepsilon v_0, u_1 + \varepsilon v_1)|_{\varepsilon=0} &= \int_{\mathbb{R}^d} [\nabla u_0 \cdot \nabla v_0 + u_1 v_1 + \mu |u_0|^{p-1} (u_0 v_0)] \\ &= \int_{\mathbb{R}^d} [u_1 v_1 - (\Delta u_0 - \mu |u_0|^{p-1} u_0) v_0] \\ &= \omega((u_1, \Delta u_0 - \mu |u_0|^{p-1} u_0), (v_0, v_1))\end{aligned}$$

Thus, the symplectic gradient of H and the associated Hamiltonian evolution equation are

$$\nabla_\omega H(u_0, u_1) = (u_1, \Delta u_0 - \mu |u_0|^{p-1} u_0), \qquad \partial_t u_0 = u_1, \quad \partial_t u_1 = \Delta u_0 - \mu |u_0|^{p-1} u_0.$$

Combining the above, we obtain the nonlinear wave equation

$$\Box u_0 = -\partial_t^2 u_0 + \Delta u_0 = -\partial_t u_1 + \Delta u_0 = \mu |u_0|^{p-1} u_0.$$

3.2. Let u and v be defined as in the problem statement. For convenience, we also define

$$z = (t, x) = (t, x_1, \dots, x_{d+1}), \qquad z' = \left(\frac{t - x_{d+1}}{2}, x_1, \dots, x_d \right), \qquad E = e^{-i(t + x_{d+1})}.$$

We can then compute

$$\begin{aligned}\partial_t v(z) &= -iEu(z') + \frac{1}{2} E \partial_t u(z'), \\ \partial_t^2 v(z) &= -Eu(z') - iE\partial_t u(z') + \frac{1}{4} E \partial_t^2 u(z'), \\ \partial_{x_{d+1}} v(z) &= -iEu(z') - \frac{1}{2} E \partial_t u(z'), \\ \partial_{x_{d+1}}^2 v(z) &= -Eu(z') + iE\partial_t u(z') + \frac{1}{4} E \partial_t^2 u(z').\end{aligned}$$

Furthermore, we define the symbols

$$\Delta_d = \sum_{k=1}^{d} \partial_{x_k}^2, \qquad \Delta_{d+1} = \sum_{k=1}^{d+1} \partial_{x_k}^2.$$

Combining all the above, we compute

$$-\partial_t^2 v(z) + \Delta_{d+1} v(z) = -\partial_t^2 v(z) + E \Delta_d u(z') + \partial_{x_{d+1}}^2 v(z) = 2iE\partial_t u(z') + E \Delta_d u(z').$$

Since u satisfies the NLS, then

$$-\partial_t^2 v(z) + \Delta_{d+1} v(z) = 2\mu E |u(z')|^{p-1} u(z') = 2\mu |Eu(z')|^{p-1} Eu(z') = 2\mu |v(z)|^{p-1} v(z).$$

Correction: If u satisfies NLS, then v satisfies NLW, with an extra factor of 2 multiplied to the power nonlinearity. This comes from the factor of $1/2$ on the Laplacian in the NLS.

3.5. [31] *Correction:* The solutions $u_{v,\lambda}$ in (3.21) of the nonperiodic focusing NLS, being Galilean transforms of rescaled soliton solutions, should be

$$u_{v,\lambda} = \lambda^{-\frac{2}{p-1}} e^{ix \cdot v} e^{-i \frac{t|v|^2}{2} + i \frac{t\tau}{\lambda^2}} Q\left(\frac{x - vt}{\lambda} \right).$$

Throughout, we will always fix the constant τ to be 1. [32]

Correction: The statement we will actually show is the following. Suppose $s < 0$ or $s < s_c$, and let $0 < \delta \ll \varepsilon \lesssim 1$. Then there exist solutions u and u' to (3.1) such that:

- At time 0, both u and u' have H^s-norm comparable to ε.

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[31] Part of the solution was inspired by [1].

[32] Since the soliton Q itself depends on τ, we fix τ a priori.

- At time 0, the H^s-separation between u' and u is comparable to δ.
- At some later time $t \lesssim \varepsilon^p$, for some positive power p depending on s, the H^s-separation between u' and u is comparable to ε.

This shows that the solution map for (3.1) is not uniformly continuous in the H^s-norm. In particular, the requirement $\delta \lesssim \varepsilon$ is mandatory, since by the triangle inequality, if solutions u and u' have H^s-norm comparable to ε at time 0, then

$$\|u'(0) - u(0)\|_{H^s_x(\mathbb{R}^d)} \le \|u'(0)\|_{H^s_x(\mathbb{R}^d)} + \|u'(0)\|_{H^s_x(\mathbb{R}^d)} \lesssim \varepsilon.$$

We begin by noting, for arbitrary $v \in \mathbb{R}^d$ and $\lambda > 0$, the identity

$$\begin{aligned}\hat{u}_{v,\lambda}(\xi) &= \lambda^{-\frac{2}{p-1}} e^{-i\frac{t|v|^2}{2}+i\frac{t}{\lambda^2}} \int_{\mathbb{R}^d} e^{ix\cdot(v-\xi)} Q\left(\frac{x-vt}{\lambda}\right) dx \\ &= \lambda^{d-\frac{2}{p-1}} e^{-i\frac{t|v|^2}{2}+i\frac{t}{\lambda^2}} \int_{\mathbb{R}^d} e^{i(\lambda x+vt)\cdot(v-\xi)} Q(x)\, dx \\ &= \lambda^{d-\frac{2}{p-1}} e^{it\left(\frac{|v|^2}{2}-v\cdot\xi+\lambda^{-2}\right)} \hat{Q}(\lambda\xi - \lambda v),\end{aligned}$$

where $\hat{u}_{v,\lambda}$ denotes the spatial Fourier transform of $u_{v,\lambda}$.

First, we consider the case

$$s_c = \frac{d}{2} - \frac{2}{p-1} > 0, \qquad 0 \le s < s_c.$$

For conciseness, we write u_λ in the place of $u_{0,\lambda}$, for any $\lambda > 0$. Note that

$$\begin{aligned}\|u_\lambda(t)\|^2_{\dot{H}^s_x(\mathbb{R}^d)} &= \lambda^{2d-\frac{4}{p-1}} \int_{\mathbb{R}^d} |\xi|^{2s} |\hat{Q}(\lambda\xi)|^2 d\xi \\ &= \lambda^{d-\frac{4}{p-1}-2s} \int_{\mathbb{R}^d} |\xi|^{2s} |\hat{Q}(\xi)|^2 d\xi \\ &= \lambda^{2(s_c-s)} \|Q\|^2_{\dot{H}^s_x(\mathbb{R}^d)}.\end{aligned}$$

In addition, fix $\lambda' > 0$, and define $\gamma = \lambda'/\lambda$.

Next, we compute the H^s-separation at time t:

$$\begin{aligned}\|u_{\lambda'}(t) - u_\lambda(t)\|^2_{\dot{H}^s_x(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} |\xi|^{2s} \left|(\lambda')^{d-\frac{2}{p-1}} e^{\frac{it}{(\lambda')^2}} \hat{Q}(\lambda'\xi) - \lambda^{d-\frac{2}{p-1}} e^{\frac{it}{\lambda^2}} \hat{Q}(\lambda\xi)\right|^2 d\xi \\ &= (\lambda')^{2(s_c-s)} \|Q\|^2_{\dot{H}^s_x(\mathbb{R}^d)} + \lambda^{2(s_c-s)} \|Q\|^2_{\dot{H}^s_x(\mathbb{R}^d)} \\ &\quad - 2(\lambda')^{d-\frac{2}{p-1}} \lambda^{d-\frac{2}{p-1}} \operatorname{Re} e^{it[(\lambda')^{-2}-\lambda^{-2}]} \int_{\mathbb{R}^d} |\xi|^{2s} \hat{Q}(\lambda'\xi) \bar{\hat{Q}}(\lambda\xi) d\xi \\ &= \lambda^{2(s_c-s)} \left[1 + \gamma^{2(s_c-s)}\right] \|Q\|^2_{\dot{H}^s_x(\mathbb{R}^d)} \\ &\quad - 2(\lambda')^{s_c+\frac{d}{2}} \lambda^{s_c-\frac{d}{2}+2s} \operatorname{Re} e^{it\frac{\lambda^2-(\lambda')^2}{\lambda^2(\lambda')^2}} \int_{\mathbb{R}^d} |\xi|^{2s} \hat{Q}(\gamma\xi) \bar{\hat{Q}}(\xi) d\xi \\ &= \lambda^{2(s_c-s)} \left\{\left[1 + \gamma^{2(s_c-s)}\right] \|Q\|^2_{\dot{H}^s_x(\mathbb{R}^d)} - 2\gamma^{s_c+\frac{d}{2}} \operatorname{Re}(e^{it\frac{\gamma^{-2}-1}{\lambda^2}} \cdot I_\gamma)\right\},\end{aligned}$$

where I_γ is the integral

$$I_\gamma = \int_{\mathbb{R}^d} |\xi|^{2s} \hat{Q}(\gamma\xi) \bar{\hat{Q}}(\xi) d\xi.$$

If $t = 0$, then by the dominated convergence theorem,

$$\lim_{\gamma\to 1} I_\gamma = \|Q\|^2_{\dot{H}^s_x(\mathbb{R}^d)}, \qquad \lim_{\gamma\to 1} \|u_{\lambda'}(0) - u_\lambda(0)\|_{\dot{H}^s_x(\mathbb{R}^d)} = 0.$$

Thus, by choosing γ to be sufficiently close to 1, we have

$$\|u_{\lambda'}(0) - u_\lambda(0)\|_{\dot{H}^s_x(\mathbb{R}^d)} \leq c \cdot \lambda^{s_c - s} \|Q\|_{\dot{H}^s_x(\mathbb{R}^d)} = c\|u_\lambda(0)\|_{\dot{H}^s_x} \simeq c\|u_{\lambda'}(0)\|_{\dot{H}^s_x},$$

for some small constant $c \ll 1$ depending on γ, but independent of λ.

Furthermore, since γ is near 1, then I_γ is almost real-valued, so by taking some

$$t \lesssim \lambda^2 \cdot |\gamma^{-2} - 1|^{-1},$$

i.e., t to be λ^2 times a large but fixed constant, the quantity $e^{it\lambda^{-2}(\gamma^{-2}-1)} I_\gamma$ becomes purely imaginary, and it follows for this choice of t that

$$\|u_{\lambda'}(t) - u_\lambda(t)\|^2_{\dot{H}^s_x(\mathbb{R}^d)} = \lambda^{2(s_c - s)} \left[1 + \gamma^{2(s_c - s)}\right] \|Q\|^2_{\dot{H}^s_x(\mathbb{R}^d)} \simeq \lambda^{2(s_c - s)} \|Q\|^2_{\dot{H}^s_x(\mathbb{R}^d)}.$$

Now, since $s \geq 0$, we have that

$$\|f\|^2_{H^s_x(\mathbb{R}^d)} \simeq \|f\|^2_{\dot{H}^s_x(\mathbb{R}^d)} + \|f\|^2_{\dot{H}^0_x(\mathbb{R}^d)}.$$

As a result, we can apply the above computations both for s and for $s = 0$. Thus, given ε and δ as in the problem statement, we can choose λ such that

$$\|u_{\lambda'}(0)\|_{H^s_x(\mathbb{R}^d)} \simeq \|u_\lambda(0)\|_{H^s_x(\mathbb{R}^d)} \simeq \varepsilon,$$
$$\|u_{\lambda'}(0) - u_\lambda(0)\|_{H^s_x(\mathbb{R}^d)} \simeq \delta \ll \varepsilon,$$
$$\|u_{\lambda'}(t) - u_\lambda(t)\|_{H^s_x(\mathbb{R}^d)} \simeq \varepsilon.$$

This completes the proof in the case $s_c > 0$ and $0 \leq s < s_c$.

It remains to consider the case $s < 0$. For this, we define the shorthand u_v for $u_{v,1}$ for any $v \in \mathbb{R}^d$. We begin by computing the H^s-norm at $t = 0$:

$$\|u_v(0)\|^2_{H^s_x(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{Q}(\xi - v)|^2 d\xi = \int_{\mathbb{R}^d} (1 + |\xi + v|^2)^s |\hat{Q}(\xi)|^2 d\xi.$$

In addition, fix $v' = (1 + \beta)v \in \mathbb{R}^3$, for sufficiently small $\beta > 0$. Moreover, we let

$$\varepsilon = \|u_v(0)\|_{H^s_x(\mathbb{R}^d)} \simeq \|u_{v'}(0)\|_{H^s_x(\mathbb{R}^d)}.$$

To obtain a possible range of values of ε, we take $|v| \geq 1$, and we write

$$\|u_v(0)\|^2_{H^s_x(\mathbb{R}^d)} = \int_{|\xi| \leq \frac{|v|}{2}} (1 + |\xi + v|^2)^s |\hat{Q}(\xi)|^2 d\xi + \int_{|\xi| \geq \frac{|v|}{2}} (1 + |\xi + v|^2)^s |\hat{Q}(\xi)|^2 d\xi = I_1 + I_2.$$

Note first that

$$I_1 \simeq |v|^{2s} \int_{\mathbb{R}^d} |\hat{Q}(\xi)|^2 d\xi \simeq |v|^{2s}.$$

Next, since Q is rapidly decreasing, so is $\hat{Q}$, hence for any $\alpha > 0$,

$$I_2 \lesssim \int_{|\xi| \geq \frac{|v|}{2}} (1 + |\xi + v|^2)^s (1 + |\xi|^2)^{-\alpha} d\xi \lesssim |v|^{-2\alpha + d}.$$

As long as the power $\alpha > 0$ is chosen to be large enough, we obtain

$$\|u_v(0)\|^2_{H^s_x(\mathbb{R}^d)} \simeq |v|^{2s}.$$

Since $s < 0$, it follows that

$$\lim_{v \to \infty} \|u_v(0)\|_{H^s_x(\mathbb{R}^d)} \to 0.$$

Thus, by continuity, we can choose ε to be any non-large constant, as desired.

For the time separation, we compute

$$\|u_{v'}(t) - u_v(t)\|^2_{H^s_x(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s \left| e^{it\left(\frac{|v'|^2}{2} - v' \cdot \xi + 1\right)} \hat{Q}(\xi - v') - e^{it\left(\frac{|v|^2}{2} - v \cdot \xi + 1\right)} \hat{Q}(\xi - v) \right|^2 d\xi$$

$$= \int_{\mathbb{R}^d} (1+|\xi|^2)^s |\hat{Q}(\xi - v')|^2 d\xi + \int_{\mathbb{R}^d} (1+|\xi|^2)^s |\hat{Q}(\xi - v)|^2 d\xi$$
$$- 2\int_{\mathbb{R}^d} (1+|\xi|^2)^s \operatorname{Re}\left\{ e^{it\left[\frac{|v'|^2-|v|^2}{2} - \beta v\cdot\xi\right]} \hat{Q}(\xi - v')\bar{\hat{Q}}(\xi - v)\right\} d\xi$$
$$= \int_{\mathbb{R}^d} (1+|\xi+v'|^2)^s |\hat{Q}(\xi)|^2 d\xi + \int_{\mathbb{R}^d} (1+|\xi+v|^2)^s |\hat{Q}(\xi)|^2 d\xi$$
$$- 2\int_{\mathbb{R}^d} (1+|\xi+v|^2)^s \operatorname{Re}\left\{ e^{it\left[\frac{(\beta^2+2\beta)|v|^2}{2} - \beta v\cdot(\xi+v)\right]} \hat{Q}(\xi - \beta v)\bar{\hat{Q}}(\xi)\right\} d\xi$$
$$= \int_{\mathbb{R}^d} (1+|\xi+v'|^2)^s |\hat{Q}(\xi)|^2 d\xi + \int_{\mathbb{R}^d} (1+|\xi+v|^2)^s |\hat{Q}(\xi)|^2 d\xi$$
$$- 2\operatorname{Re} e^{it\frac{\beta^2|v|^2}{2}} \int_{\mathbb{R}^d} (1+|\xi+v|^2)^s e^{it\beta v\cdot\xi} \hat{Q}(\xi - \beta v)\bar{\hat{Q}}(\xi) d\xi$$
$$= \int_{\mathbb{R}^d} (1+|\xi+v'|^2)^s |\hat{Q}(\xi)|^2 d\xi + \int_{\mathbb{R}^d} (1+|\xi+v|^2)^s |\hat{Q}(\xi)|^2 d\xi - 2J_{t,\beta},$$

where

$$J_{t,\beta} = \operatorname{Re} e^{it\frac{\beta^2|v|^2}{2}} \int_{\mathbb{R}^d} (1+|\xi+v|^2)^s e^{it\beta v\cdot\xi} \hat{Q}(\xi - \beta v)\bar{\hat{Q}}(\xi) d\xi.$$

By the dominated convergence theorem, we have

$$\lim_{\beta\searrow 0} J_{0,\beta} = \int_{\mathbb{R}^d} (1+|\xi+v|^2)^s |\hat{Q}(\xi)|^2 d\xi, \qquad \lim_{\beta\searrow 0} \|u_{v'}(0) - u_v(0)\|_{H^s_x(\mathbb{R}^d)} = 0.$$

Thus, by choosing β to be sufficiently small, we obtain for some small $c \ll 1$ that

$$\|u_{v'}(0) - u_v(0)\|^2_{H^s_x(\mathbb{R}^d)} \le c^2 \int_{\mathbb{R}^d} (1+|\xi+v|^2)^s |\hat{Q}(\xi)|^2 d\xi = c^2 \|u_v(0)\|^2_{H^s_x(\mathbb{R}^d)} = c^2\varepsilon^2.$$

Next, take $t = \beta^{-2}|v|^{-1}$, i.e., t a large constant times $|v|^{-1} \simeq \varepsilon^{-1/s}$. Choosing a component $1 \le m \le d$ of v such that $|v_m| \simeq |v|$, then we can bound

$$|J_{t,\beta}| \lesssim \left| \int_{\mathbb{R}^d} (1+|\xi+v|^2)^s \hat{Q}(\xi - \beta v)\bar{\hat{Q}}(\xi) \cdot \frac{1}{\beta^{-1} v_m |v|^{-1}} \partial_m e^{i\beta^{-1} v|v|^{-1}\cdot\xi} d\xi \right|$$
$$\lesssim \beta \int_{\mathbb{R}^d} |\partial_m[(1+|\xi+v|^2)^s \hat{Q}(\xi - \beta v)\bar{\hat{Q}}(\xi)]| d\xi,$$

where we integrated by parts in the last step. Recalling that $\hat{Q}$ is rapidly decreasing, we can, using the same techniques as before, derive the bound

$$|J_{t,\beta}| \lesssim \beta |v|^{2s} \simeq \beta\varepsilon.$$

As a result, with β sufficiently small, and with this choice of $t \simeq \varepsilon^{-1/s}$, we have

$$\|u_{v'}(t) - u_v(t)\|^2_{H^s_x(\mathbb{R}^d)} \simeq \varepsilon^2, \qquad \|u_{v'}(t) - u_v(t)\|_{H^s_x(\mathbb{R}^d)} \simeq \varepsilon.$$

Recall that, with $\delta = c\varepsilon$, we also had

$$\|u_{v'}(0)\|_{H^s_x(\mathbb{R}^d)} \simeq \|u_v(0)\|_{H^s_x(\mathbb{R}^d)} \simeq \varepsilon,$$
$$\|u_{v'}(0) - u_v(0)\|_{H^s_x(\mathbb{R}^d)} \simeq \delta \ll \varepsilon.$$

This completes the proof for the case $s < 0$.

3.6. In general, (3.2) has the conjugation invariance property: if u is a classical solution of (3.2), then its conjugate $\bar{u}$ also solves (3.2). To see this, we simply compute:

$$\Box\bar{u} + \Delta\bar{u} - \mu|\bar{u}|^{p-1}\bar{u} = \overline{\Box u + \Delta u - \mu|u|^{p-1}u} \equiv 0.$$

Correction: We will prove the following: if u is a classical solution of (3.2), and if both $u(t_0)$ and $\partial_t u(t_0)$ are real-valued, then u is everywhere real-valued. Note that the additional condition on $\partial_t u(t_0)$ is necessary, since if $u(t_0)$ and $\partial_t u(t_0)$ are purely real and imaginary, respectively, then u cannot be everywhere real at a time near t_0.

If u is as above, then $\bar{u}$ is also a classical solution of (3.2), and at t_0, it satisfies

$$\bar{u}(t_0) = u(t_0), \qquad \partial_t\bar{u}(t_0) = \overline{\partial_t u}(t_0) = \partial_t u(t_0),$$

since both $u(t_0)$ and $\partial_t u(t_0)$ are real-valued. Thus, by uniqueness (Proposition 3.3), it follows that u and $\bar{u}$ are everywhere equal, and hence u is everywhere real-valued.

3.7. Let $R \in SO(d, \mathbb{R})$ denote an arbitrary spatial rotation. Suppose u and v are classical solutions to (3.1) and (3.2), respectively. By spatial rotation symmetry, the functions

$$u_R, v_R : I \times \mathbb{R}^d \to \mathbb{C}, \qquad u_R(t,x) = u(t,Rx), \qquad v_R(t,x) = v(t,Rx)$$

are also classical solutions to (3.1) and (3.2), respectively.

Now, suppose $u(t_0)$ is spherically symmetric. Then, u_R solves (3.1), and $u_R(t_0) = u(t_0)$. By uniqueness (see Proposition 3.2), it follows that $u_R = u$ everywhere. Since this is true for any rotation R, then u is spherically symmetric.

Likewise, if $v[t_0] = (v(t_0), \partial_t v(t_0))$ is spherically symmetric, then v_R solves (3.2), and $v_R[t_0] = v[t_0]$. By uniqueness (Proposition 3.3), $v_R = v$ everywhere. By varying over all rotations R, it follows that v is spherically symmetric.

3.9. *Correction:* In this problem, we are considering the *focusing* NLW.

Fix $t_0 > 0$, and consider the solution to the focusing NLW in (3.6):

$$u(t,x) = c_p(t_0 - t)^{\frac{-2}{p-1}}, \qquad c_p = \left[\frac{2(p+1)}{(p-1)^2}\right]^{\frac{1}{p-1}}.$$

This is a smooth solution that blows up at time t_0. Let $\varphi : \mathbb{R}^d \to [0,1]$ be a smooth and compactly supported cutoff function that is identically 1 on the ball about the origin of radius R, with $R \gg t_0$. Consider the compactly supported initial data $(u_0, u_1) = (\varphi u(0), \varphi\partial_t u(0))$, which we impose at time $t = 0$. Solving the focusing NLW with this data yields a classical solution v. [33] By uniqueness and finite speed of propagation (Proposition 3.3), it follows that u and v must coincide on a cylinder $C = \{(t,x) \mid 0 \leq t < t_0, |x| < r\}$, for some $r > 0$. Since u blows up at $t = t_0$ on C, then v blows up at $t = t_0$ on C as well.

3.10. Since u is a strong H^s-solution to (3.1), with data $u(t_0) = u_0$, we have, by definition,

$$u \in C^0_{t,\mathrm{loc}} H^s_x(I \times \mathbb{R}^d), \qquad u(t) = e^{\frac{1}{2}i(t-t_0)\Delta}u_0 - i\mu\int_{t_0}^t e^{\frac{1}{2}i(t-\tau)\Delta}[|u(\tau)|^{p-1}u(\tau)]d\tau$$

for any $t \in I$. We can restate the above in terms of t_1 rather than t_0: [34]

$$u(t) = e^{\frac{1}{2}i(t-t_1)\Delta}e^{\frac{1}{2}i(t_1-t_0)\Delta}u_0 - i\mu\int_{t_1}^t e^{\frac{1}{2}i(t-\tau)\Delta}[|u(\tau)|^{p-1}u(\tau)]d\tau$$

[33] Here, we must apply existence theorems for classical solutions of the NLW.

[34] Note that in the last term below, we implicitly used that the strong solution u is continuous in time in order to factor the linear propagator $e^{i(t-t_1)\Delta/2}$ out of the time integral.

$$- e^{\frac{1}{2}i(t-t_1)\Delta}\left\{i\mu\int_{t_0}^{t_1} e^{\frac{1}{2}i(t_1-\tau)\Delta}[|u(\tau)|^{p-1}u(\tau)]d\tau\right\}.$$

Since $u_1 = u(t_1)$, and since u is a strong H^s-solution to (3.1) with data u_0, then

$$u_1 = u^{\frac{1}{2}i(t_1-t_0)\Delta}u_0 - i\mu\int_{t_0}^{t_1} e^{\frac{1}{2}i(t-\tau)\Delta}[|u(\tau)|^{p-1}u(\tau)]d\tau.$$

Consequently, we obtain, as desired

$$u(t) = e^{\frac{1}{2}i(t-t_1)\Delta}u_1 - i\mu\int_{t_1}^{t} e^{\frac{1}{2}i(t-\tau)\Delta}[|u(\tau)|^{p-1}u(\tau)]d\tau,$$

and it follows that u is a strong H^s-solution to (3.1), with data $u(t_1) = u_1$.

Next, since conjugation preserves the H^s-norm, the function

$$t \mapsto \tilde{u}(t) = \overline{u(-t)}, \qquad -t \in I$$

is also a continuous map into H^s. Moreover, by definition, for any such t, we have

$$\tilde{u}(t) = \overline{e^{\frac{1}{2}i(-t-t_0)\Delta}u(t_0)} + i\mu\int_{t_0}^{-t} \overline{e^{\frac{1}{2}i(-t-\tau)\Delta}[|u(\tau)|^{p-1}u(\tau)]}d\tau.$$

Since the linear Schrödinger equation is conjugation-invariant, then

$$\begin{aligned}
\tilde{u}(t) &= e^{\frac{1}{2}i(t+t_0)\Delta}\overline{u(t_0)} + i\mu\int_{t_0}^{-t} e^{\frac{1}{2}i(t+\tau)\Delta}\overline{[|u(\tau)|^{p-1}u(\tau)]}d\tau \\
&= e^{\frac{1}{2}i(t+t_0)\Delta}[\tilde{u}(-t_0)] - i\mu\int_{-t_0}^{t} e^{\frac{1}{2}i(t-\tau)\Delta}[|\bar{u}(-\tau)|^{p-1}\bar{u}(-\tau)]d\tau \\
&= e^{\frac{1}{2}i(t+t_0)\Delta}\bar{u}_0 - i\mu\int_{-t_0}^{t} e^{\frac{1}{2}i(t-\tau)\Delta}[|\tilde{u}(\tau)|^{p-1}\tilde{u}(\tau)]d\tau.
\end{aligned}$$

The above equation shows that $\tilde{u}$ is a strong H^s-solution to (3.1), with data $\tilde{u}(-t_0) = \bar{u}_0$.

Finally, let v be a strong H^s-solution to (3.2) on an interval I, with initial datum

$$v[t_0] = (v(t_0), \partial_t v(t_0)) = (v_0, v_0'), \qquad t_0 \in I.$$

As before, fix another time $t_1 \in I$, and let

$$v[t_1] = (v(t_1), \partial_t v(t_1)) = (v_1, v_1').$$

For simplicity, we write

$$\begin{aligned}
L(s)(f,g) &= \cos(s\sqrt{-\Delta})f + \frac{\sin(s\sqrt{-\Delta})}{\sqrt{-\Delta}}g, \\
\mathcal{L}(s)(f,g) &= (L(s)(f,g), \partial_s[L(s)(f,g)]).
\end{aligned}$$

representing the linear propagator for the wave equation, written as a first-order system. Applying the semigroup property for $\mathcal{L}$, the proof proceeds like in the case of the NLS:

$$\begin{aligned}
(v(t), \partial_t v(t)) &= \mathcal{L}(t-t_0)(v_0, v_0') - \mu\int_{t_0}^{t} \mathcal{L}(t-\tau)(0, |v(\tau)|^{p-1}v(\tau))d\tau \\
&= \mathcal{L}(t-t_1)[\mathcal{L}(t_1-t_0)(v_0, v_0')] - \mu\int_{t_1}^{t} \mathcal{L}(t-\tau)(0, |v(\tau)|^{p-1}v(\tau)])d\tau \\
&\qquad \mathcal{L}(t-t_1)\left\{i\mu\int_{t_0}^{t_1} \mathcal{L}(t_1-\tau)(0, |v(\tau)|^{p-1}v(\tau))d\tau\right\} \\
&= \mathcal{L}(t-t_1)(v_1, v_1') - \mu\int_{t_1}^{t} \mathcal{L}(t-\tau)(0, |v(\tau)|^{p-1}v(\tau)])d\tau.
\end{aligned}$$

Thus, v is a strong H^s-solution to (3.2), with data $v[t_1] = (v_1, v_1')$.

3.12. Suppose u is a weak H^s-solution to (3.1) with data $u(t_0) = u_0$, where $s > d/2$, i.e.,

$$u \in L_t^\infty H_x^s(I \times \mathbb{R}^d), \qquad u(t) = e^{\frac{1}{2}i(t-t_0)\Delta}u_0 - i\mu \int_{t_0}^t e^{\frac{1}{2}i(t-\tau)\Delta}[|u(\tau)|^{p-1}u(\tau)]d\tau,$$

with the integral equation holding almost everywhere. Fixing two nearby times $t, t' \in I$ for which the above integral equation holds, then we can bound

$$\begin{aligned}
\|u(t') - u(t)\|_{H_x^s} &\leq \|[e^{\frac{1}{2}i(t'-t_0)\Delta} - e^{\frac{1}{2}i(t-t_0)\Delta}]u_0\|_{H_x^s} \\
&\quad + \left\| \int_{t_0}^{t'} e^{\frac{1}{2}i(t'-\tau)\Delta}[|u(\tau)|^{p-1}u(\tau)]d\tau - \int_{t_0}^{t} e^{\frac{1}{2}i(t-\tau)\Delta}[|u(\tau)|^{p-1}u(\tau)]d\tau \right\|_{H_x^s} \\
&\leq \|[e^{\frac{1}{2}i(t'-t_0)\Delta} - e^{\frac{1}{2}i(t-t_0)\Delta}]u_0\|_{H_x^s} + \left\| \int_t^{t'} e^{\frac{1}{2}i(t-\tau)\Delta}[|u(\tau)|^{p-1}u(\tau)]d\tau \right\|_{H_x^s} \\
&\quad + \left\| [e^{\frac{1}{2}i(t'-t)\Delta} - 1] \int_{t_0}^{t'} e^{\frac{1}{2}i(t-\tau)\Delta}[|u(\tau)|^{p-1}u(\tau)]d\tau \right\|_{H_x^s} \\
&= L + N_1 + N_2.
\end{aligned}$$

By the dominated convergence theorem, the linear part L satisfies

$$L^2 = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s \left| e^{i(t'-t_0)|\xi|^2} - e^{i(t-t_0)|\xi|^2} \right|^2 |\hat{u}_0(\xi)|^2 d\xi \to 0$$

as $t' \to t$, hence it is continuous in time. For the nonlinear part N_1, we use Lemma A.8, in particular (A.18), along with the fact that the $u(\tau)$'s are uniformly bounded in H^s:

$$N_1 \lesssim \int_t^{t'} \|u(\tau)\|_{H_x^s}^p d\tau \leq (t' - t)\|u\|_{L_t^\infty H_x^s}^p.$$

In particular, the right-hand side goes to 0 as $t' \to t$. Similarly, for the remaining term N_2, letting $\mathcal{F}$ denote the Fourier transform in the spatial variables, then

$$\begin{aligned}
N_2 &= \left[\int_{\mathbb{R}^d} (1 + |\xi|^2)^s \left| e^{i(t'-t)|\xi|^2} - 1 \right|^2 \left| \int_{t_0}^{t'} \mathcal{F}\{e^{\frac{1}{2}i(t-\tau)\Delta}[|u(\tau)|^{p-1}u(\tau)]\} d\tau \right|^2 d\xi \right]^{\frac{1}{2}} \\
&\leq \int_{t_0}^{t'} \left[\int_{\mathbb{R}^d} (1 + |\xi|^2)^s \left| e^{i(t'-t)|\xi|^2} - 1 \right|^2 \left| \mathcal{F}\{e^{\frac{1}{2}i(t-\tau)\Delta}[|u(\tau)|^{p-1}u(\tau)]\} \right|^2 d\xi \right]^{\frac{1}{2}} d\tau \\
&\leq (t' - t_0)^{\frac{1}{2}} \left[\int_{t_0}^{t'} \int_{\mathbb{R}^d} (1 + |\xi|^2)^s \left| e^{i(t'-t)|\xi|^2} - 1 \right|^2 \left| \mathcal{F}\{e^{\frac{1}{2}i(t-\tau)\Delta}[|u(\tau)|^{p-1}u(\tau)]\} \right|^2 d\xi d\tau \right]^{\frac{1}{2}}.
\end{aligned}$$

By Lemma A.8, there is some constant $C > 0$, independent of τ, such that

$$\int_{\mathbb{R}^d} (1 + |\xi|^2)^s \mathcal{F}\{e^{\frac{1}{2}i(t-\tau)\Delta}[|u(\tau)|^{p-1}u(\tau)]\} d\xi \lesssim \|u\|_{L_t^\infty H_x^s}^p < C.$$

Thus, by the dominated convergence theorem, it follows that $N_2 \to 0$ as $t' \to t$.

Consequently, u can be considered (by replacing a subset of measure zero) as a continuous function into $H^s(\mathbb{R}^d)$, hence u is a strong H^s-solution.

3.13. Let J be any time interval containing t_0, and let u, v be two strong solutions to (3.1) on J with the same initial data at t_0. Define the subset

$$A = \{t \in J \mid u(t) = v(t)\}.$$

Since both u and v are continuous with respect to t, then A is closed. Furthermore, if $s \in A$, then by the local uniqueness assumption, there is an open interval I containing s such that $u(t) = v(t)$ for any $t \in I$. As a result, $I \subseteq A$, and it follows that A is open. Since A is open, closed, and nonempty (since $t_0 \in A$ by assumption), it follows from connectedness considerations that $A = J$. Thus, u and v coincide everywhere on J.

3.14. First, note that for any $x \in \mathbb{R}^d$, we have, by definition and induction, the inequality

$$|\nabla^j \langle x \rangle^k| \lesssim_{d,k} \langle x \rangle^{k-j},$$

for any nonnegative integers $0 \leq j \leq k$, where ∇ denotes the spatial gradient on $\mathbb{R}^d$. As a result, we can apply the above in conjunction with the Leibniz rule in order to obtain

$$\|f\|_{H^{k,k}_x(\mathbb{R}^d)} = \sum_{j=0}^{k} \|\langle x \rangle^j f\|_{H^{k-j}_x(\mathbb{R}^d)} \lesssim \sum_{a+b \leq k} \|\nabla^b(\langle x \rangle^a f)\|_{L^2_x(\mathbb{R}^d)} \lesssim \sum_{a+b \leq k} \|\langle x \rangle^a \nabla^b f\|_{L^2_x(\mathbb{R}^d)}.$$

Similarly, again by the above pointwise inequality and the Leibniz rule, we have

$$\sum_{a+b \leq k} \|\langle x \rangle^a \nabla^b f\|_{L^2_x(\mathbb{R}^d)} \lesssim \sum_{a+b \leq k} \|\nabla^b(\langle x \rangle^a f)\|_{L^2_x(\mathbb{R}^d)} \lesssim \|f\|_{H^{k,k}_x(\mathbb{R}^d)}.$$

Using the above equivalent formulation of the $H^{k,k}$-norm and Hölder's inequality yields

$$\begin{aligned} \|fg\|_{H^{k,k}_x(\mathbb{R}^d)} &\lesssim \sum_{a+b+c+p=k} \|\langle x \rangle^a \nabla^b f \nabla^c g\|_{L^2_x(\mathbb{R}^d)} \\ &\lesssim \sum_{a+b+c+p=k} \|\langle x \rangle^a \nabla^b f\|_{L_x^{\frac{2k}{a+b}}(\mathbb{R}^d)} \|\nabla^c f\|_{L_x^{\frac{2k}{c+p}}(\mathbb{R}^d)}. \end{aligned}$$

Given a, b, c, p as in the terms on the right-hand side above, since $k > d/2$, we have

$$\frac{d}{2}\left(1 - \frac{a}{k} - \frac{b}{k}\right) < k - a - b, \qquad \frac{d}{2}\left(1 - \frac{c}{k} - \frac{p}{k}\right) < k - c - p.$$

Thus, the Gagliardo-Nirenberg inequality (Proposition A.3) yields

$$\|fg\|_{H^{k,k}_x(\mathbb{R}^d)} \lesssim \sum_{a+b+c+p=k} \|\langle x \rangle^a \nabla^b f\|_{H^{k-a-b}_x(\mathbb{R}^d)} \|\nabla^c f\|_{H^{k-c-p}_x(\mathbb{R}^d)}.$$

Finally, returning to our pointwise inequality, we have, as desired,

$$\|fg\|_{H^{k,k}_x(\mathbb{R}^d)} \lesssim \sum_{a+b \leq k} \|\langle x \rangle^a \nabla^b f\|_{L^2_x(\mathbb{R}^d)} \sum_{c+p \leq k} \|\nabla^c f\|_{L^2_x(\mathbb{R}^d)} \lesssim \|f\|_{H^{k,k}_x(\mathbb{R}^d)} \|g\|_{H^{k,k}_x(\mathbb{R}^d)}.$$

Appendix A: Tools from Harmonic Analysis

A.15. *Correction:* For the first inequality, we need an extra condition, e.g., u having zero mean. Otherwise, we can consider a constant function $u \equiv c \neq 0$, for which we have

$$\|u\|_{L^\infty(I)} = c \neq 0, \qquad \|u\|^{1/2}_{L^2(I)} \|\partial_t u\|^{1/2}_{L^2(I)} = 0.$$

With the extra mean-free assumption for u, we can conclude via the intermediate value theorem that $u(t_0) = 0$ for some $t_0 \in I$. [35] By the fundamental theorem of calculus,

$$|u(t)|^2 = \int_{t_0}^{t} \partial_t |u|^2 \lesssim \int_{t_0}^{t} |u||\partial_t u| \lesssim \|u\|_{L^2(I)} \|\partial_t u\|_{L^2(I)}, \qquad t \in I.$$

This proves the Gagliardo-Nirenberg inequality.

Next, suppose $I = [a, b]$, and let $\underline{u}$ denote the mean of u on I. As a first step, we assume that $\underline{u} = 0$. Integrating by parts, then we obtain

$$\begin{aligned} \int_a^b |u|^2 &= \int_a^b \left[u(t) \cdot \partial_t \int_a^t u \right] dt \\ &= u(b) \cdot \int_a^b u - \int_a^b \left[\partial_t u(t) \cdot \int_a^t u \right] dt \\ &= - \int_a^b \left[\partial_t u(t) \cdot \int_a^t u \right] dt. \end{aligned}$$

Applying Hölder's inequality yields

$$\begin{aligned} \|u\|_{L^2(I)}^2 &\leq \|\partial_t u\|_{L^2(I)} \left[\int_a^b \left(\int_a^t u \right)^2 dt \right]^{\frac{1}{2}} \\ &\leq \|\partial_t u\|_{L^2(I)} \left(\int_a^b |I| \|u\|_{L^2(I)}^2 \right)^{\frac{1}{2}} \\ &\leq |I| \|\partial_t u\|_{L^2(I)} \|u\|_{L^2(I)}, \end{aligned}$$

which implies the Poincaré inequality in the case $\underline{u} = 0$.

Finally, for general u, since $u - \underline{u}$ has zero mean, then

$$\|u - \underline{u}\|_{L^2(I)} \leq |I| \|\partial_t (u - \underline{u})\|_{L^2(I)} = |I| \|\partial_t u\|_{L^2(I)}.$$

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[35] By a standard limiting argument, we can always assume that u is smooth on the interior of I.